Extra Credit (15 points)

Math 365, Spring 2015

Due April ???

1. Consider the temperature problem from Homeworks #2 and #3. We modeled this as a *2-point boundary value problem* (BVP), given by

$$\begin{aligned} T''(x) &= f(x) \\ T(0) &= T_0 \\ T(L) &= T_N \end{aligned} \tag{1}$$

where $T(x)$ represents our temperature K°, T_0 and T_N are temperatures we impose at the endpoints of the rod, and $f(x)$ is a heat source. In steady state, the flux of heat in or out of the rod should be balanced exactly by the heat that the flame is adding to the rod. Mathematically, we can express this idea as

$$T'(L) - T'(0) = \int_0^L f(x)dx. \tag{2}$$

It turns out that this property holds in our discrete equations as well. We discretize the boundary value problem using the finite difference approximation

$$\frac{T_{j-1} - 2T_j + T_{j+1}}{h^2} = f(x_j), \qquad j = 1, 2, 3, \ldots, N-1$$

where T_j is an approximation to the temperature $T(x_j)$ at a discrete set of points $x_j = jh$, $h = L/N$. We can represent this discretization using the linear system

$$A\mathbf{T} = \mathbf{f} - \mathbf{b}$$

where A is an $(N-1) \times (N-1)$ matrix, and $\mathbf{T}$, $\mathbf{f}$ and $\mathbf{b}$ are $N-1$ vectors. The vector $\mathbf{b}$ contains our boundary conditions T_0 and T_N.

We can then approximate the derivatives (or *fluxes*) of $T(x)$ at the endpoints as

$$\begin{aligned} T'(0) &\approx \frac{T_1 - T_0}{h} \\ T'(L) &\approx \frac{T_N - T_{N-1}}{h} \end{aligned}$$

where T_0 and T_N are the end point temperatures that we imposed.

Show the following.

(a) Use the Fundamental Theorem of Calculus to show that (2) holds for the 2-point BVP in (1).

(b) We can approximate the integral in (2) using a *Riemann* sum and write

$$\int_0^L f(x)dx \approx h \sum_{j=1}^{N-1} f(x_j)$$

Write a Matlab script to verify numerically that the discrete balance law

$$\frac{T_N - T_{N-1}}{h} - \frac{T_1 - T_0}{h} = h \sum_{j=1}^{N-1} (A\mathbf{T} + \mathbf{b}) = h \sum_{j=1}^{N-1} f(x_j). \tag{3}$$

holds for the temperature problem in Homework # 2. **Note :** You will have to do a bit of pencil and paper work to get the problem from Homework # 2 into the correct form for (3). Each of the terms in (3) should equal `-443.1134627263788843`.

(c) Show (using pencil and paper) that (3) holds for a general $f(x)$.