Piecewise Polynomial Interpolation

Piecewise constant interpolation

The easiest way to interpolate data is to do a piecewise constant interpolation.

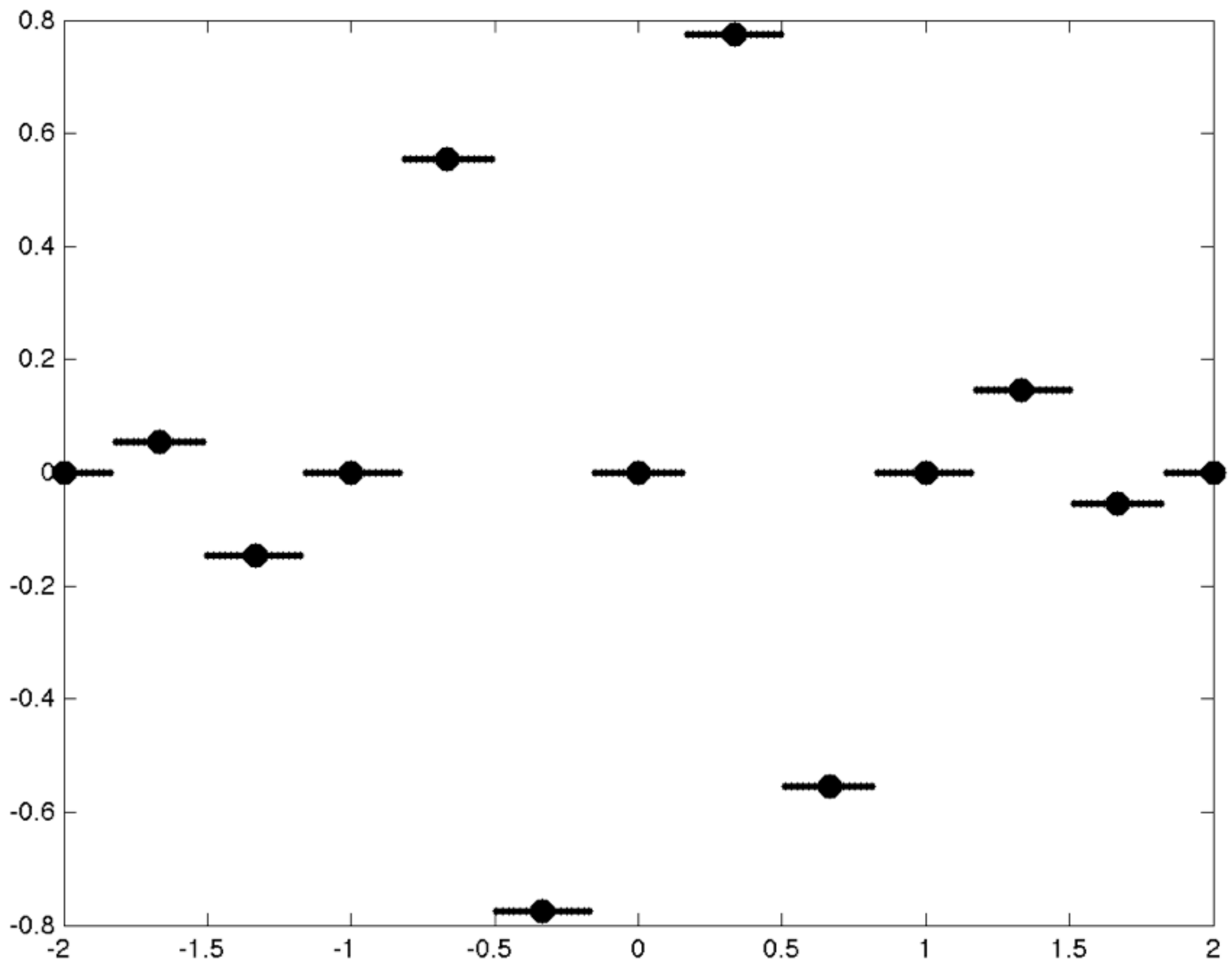
We could choose the left endpoint of the interval:

$$P(x) = y_i, \quad x \in [x_i, x_{i+1}]$$

or the nearest point:

$$P(x) = y_i, \qquad x \in \left[\frac{x_{i-1} + x_i}{2}, \frac{x_i + x_{i+1}}{2}\right]$$

Piecewise constant interpolation



Choose the nearest value

Piecewise linear interpolation

Suppose we have data point $(x_i, f(x_i))$, $i = 1, 2, \ldots N$. A piecewise linear polynomial that interpolates these points is given by

$$P(x) = P_i(x), \quad x \in [x_i, x_{i+1}]$$

where the polynomials $P_i(x)$ can be written as

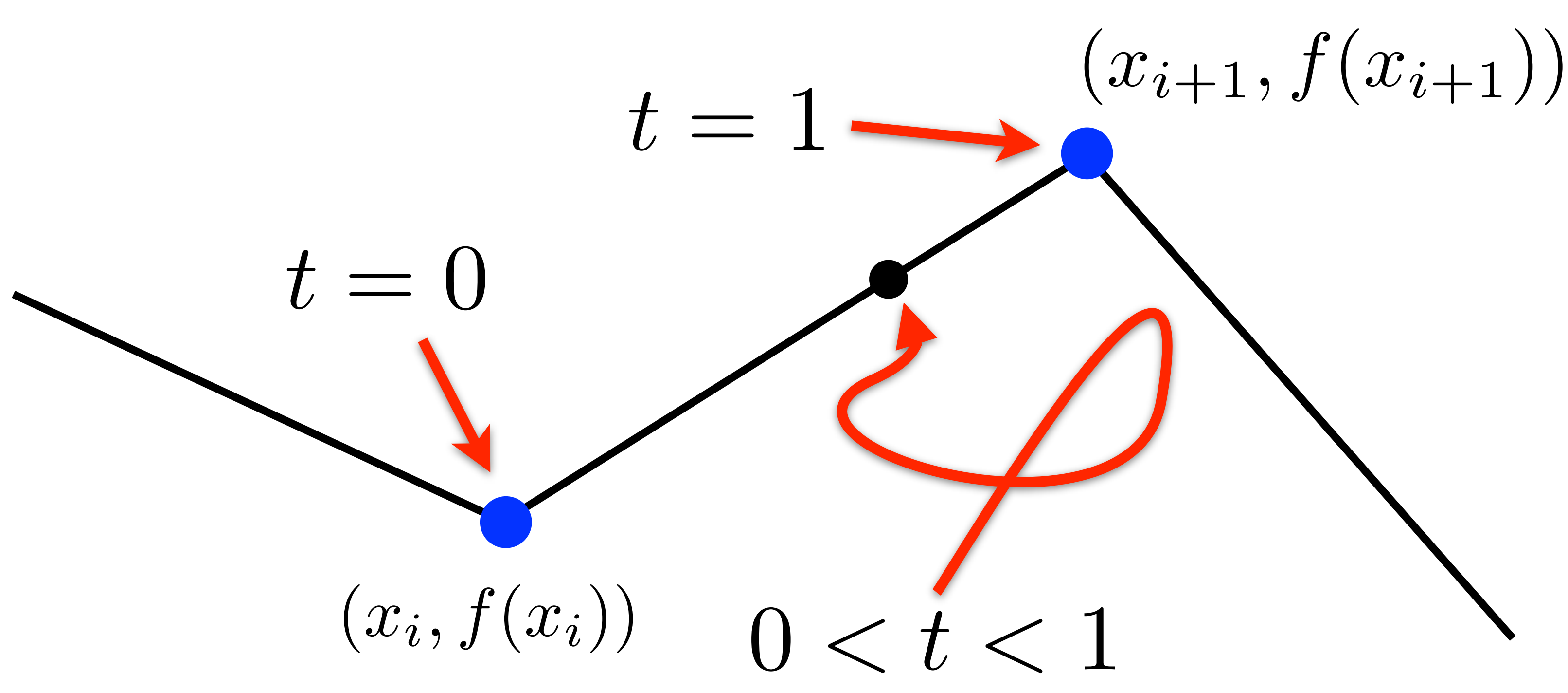
$$P_i(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)$$

Check : $P_i(x_i) = f(x_i)$ and $P_{i+1}(x_{i+1}) = f(x_{i+1})$. This polynomial interpolates the data points.

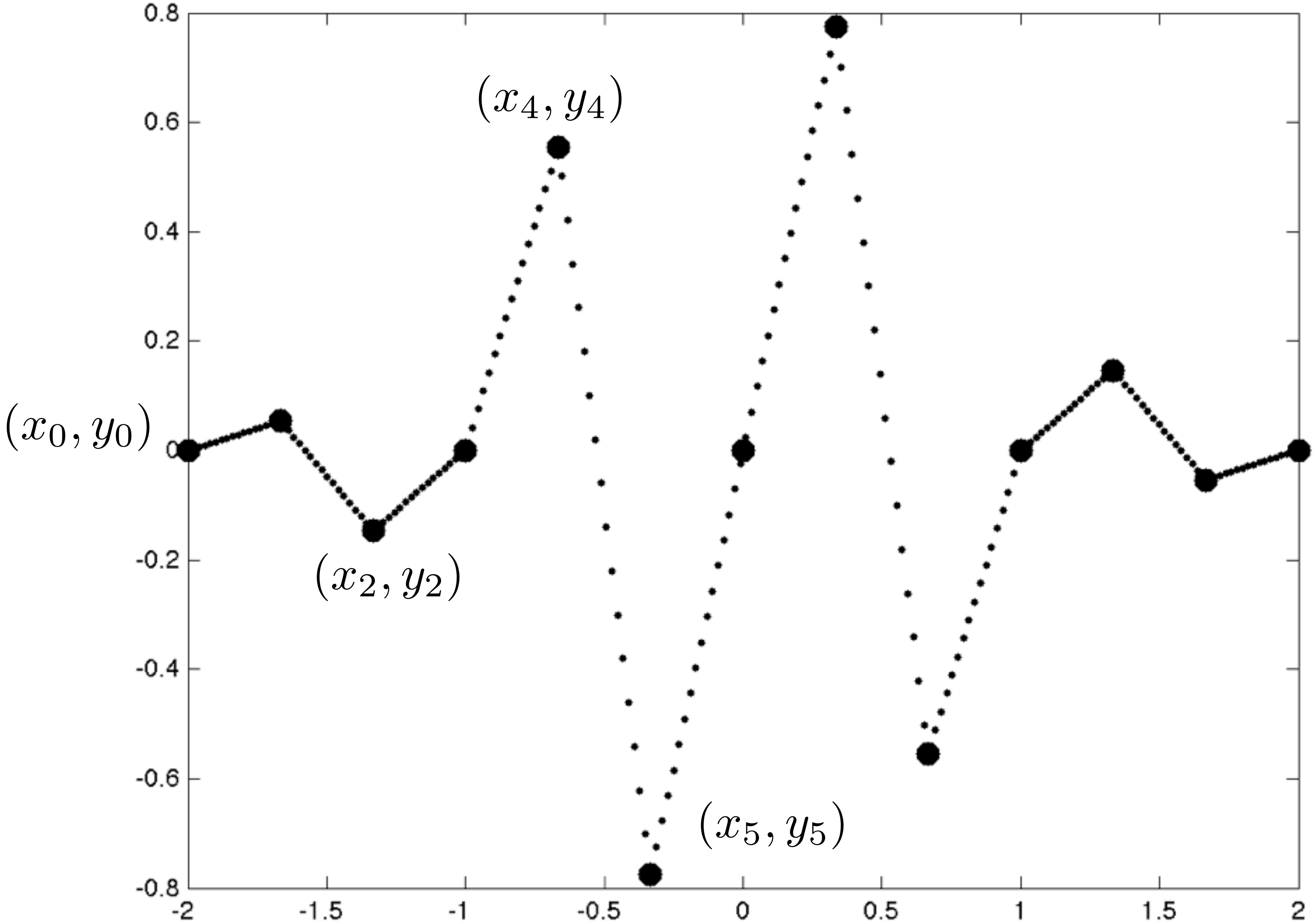
Piecewise linear interpolation

We can also write this as a function of a local variable t which measures the distance along the interval.

$$P_i(x) = (1-t)f(x_i) + tf(x_{i+1}), \quad t = \frac{x - x_i}{x_{i+1} - x_i} \in [0,1]$$



Piecewise linear interpolation



Piecewise linear interpolation

Piecewise linear interpolation has these properties

- *Continuous* at nodes (x_i, y_i).
- Derivatives are not continuous (the function is not smooth!)

Can we come design a method that matches derivatives at the nodes as well?

Piecewise cubic Hermite interpolation

Suppose we want to match both the function values and the derivatives.

$$\begin{aligned} P_i(x) &= \left[1 - t^2(3-2t)\right] f(x_i) + \left[t^2(3-2t)\right] f(x_{i+1}) \\ &+ \left[t(t-1)^2\right] h f'(x_i) + \left[t^2(t-1)\right] h f'(x_{i+1}) \end{aligned}$$

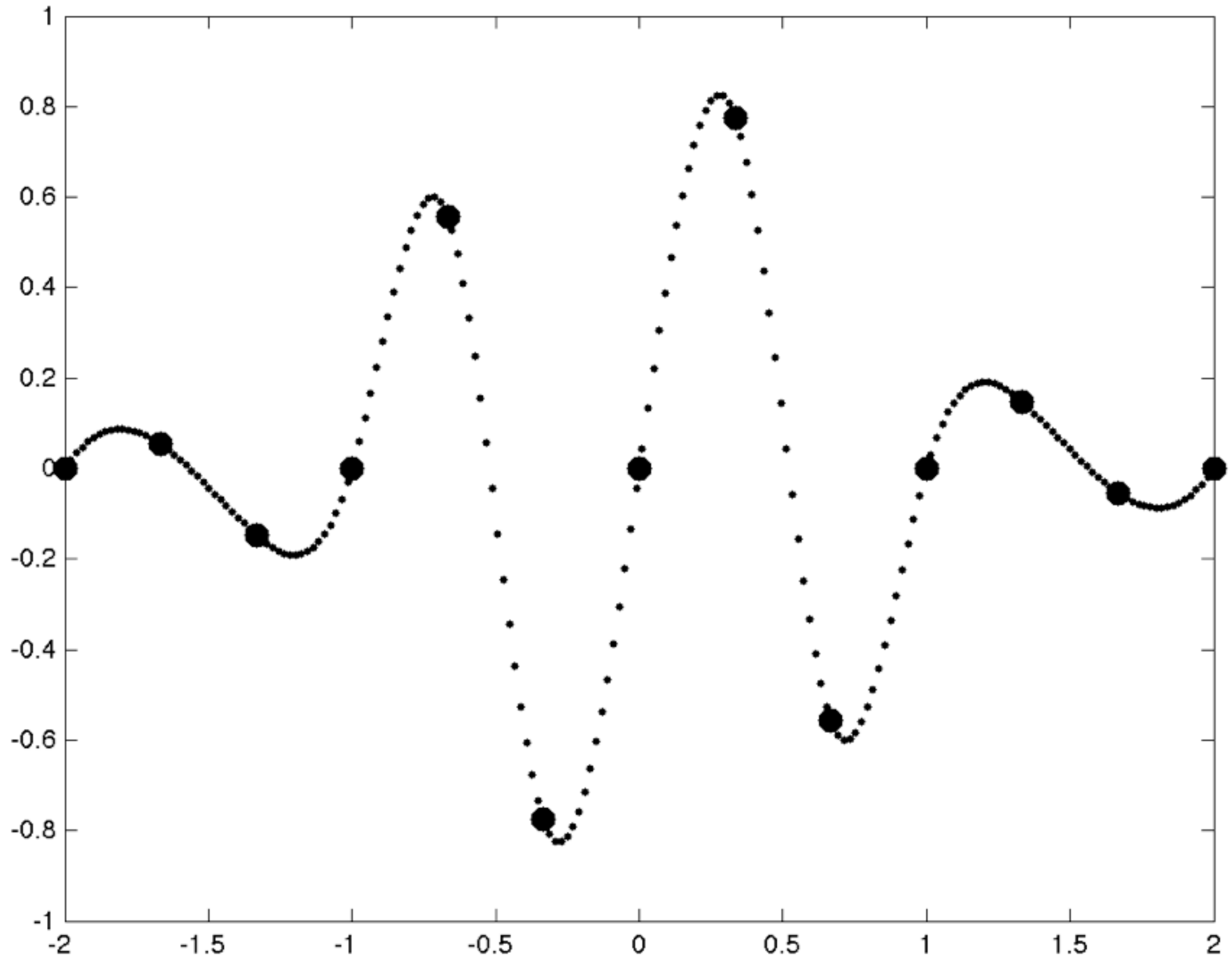
where

$$t = \frac{x - x_i}{x_{i+1} - x_i} \in [0, 1]$$

and $h = x_{i+1} - x_i$.

Note that this assumes that we can evaluate the derivative function $f'(x)$.

Piecewise linear interpolation



9