Introduction to Matrices

Writing a linear system using matrices :

$$2x + 4y - 2z = 2$$
$$4x + 9y - 3z = 8$$
$$-2x - 3y + 7z = 10$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

What we now want to do is investigate how we can solve this linear system.

Matrix notation

The entries of matrix A are given by a_{ij}, where i is the row, and j is the column

$a_{21} = 4 \qquad a_{12} = 6 \qquad a_{23} = 11 \qquad a_{33} = 7$

$$A = \begin{bmatrix} 2 & 6 & 1 \\ 4 & 9 & 11 \\ -2 & -3 & 7 \end{bmatrix}$$

$a_{21} =$ "down 2 and over 1" $\qquad a_{23} =$ "down 2 and over 3"

$a_{12} =$ "down 1 and over 2" $\qquad a_{33} =$ "down 3 and over 3"

Matrix addition

To add two matrices which have the same shape, we simply add their components

Example :

$$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix}$$

Multiplication by a scalar also happens componentwise :

$$3 \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 12 & -3 \end{bmatrix}$$

Matrix multiplication

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -2 \\ -1 & -3 & 7 \end{bmatrix} \begin{bmatrix} 2 & 6 & 1 \\ 4 & 9 & 11 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 15 \\ 12 & 27 & -1 \\ -28 & -54 & 15 \end{bmatrix}$$

We can describe each entry as a *dot product* of a row with a column vector

$$a_{ij} = (\textbf{Row } i) \cdot (\textbf{Column } j)$$

Example :

$$a_{31} = (-1, -3, 7) \cdot (2, 4, -2) = (-1)(2) + (-3)(4) + (7)(-2) = -28$$

Laws for Matrix Operations

For matrices A and B which have the same shape (assume square for now), we have

$$\begin{aligned} A + B &= B + A \\ c(A + B) &= cA + cB \\ A + (B + C) &= (A + B) + C \end{aligned}$$

where c is a scalar.

Laws for matrix operations

For matrices A, B, and C we have the following

$$\begin{aligned} C(A+B) &= CA + CB \\ (A+B)C &= AC + BC \end{aligned}$$

$$A(BC) = (AB)C$$

But we do not have a "commutative" law

$$AB \neq BA \qquad \text{in general}$$

Matrix multiplication

Multiplication on the right by a vector can be thought of a forming a linear combination of *columns.*

$$\begin{bmatrix} 2 & 4 & 5 \\ 1 & 9 & -3 \\ -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} =$$

Result is a column vector

$$a \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + b \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} + c \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$$

Multiplication on the right can be thought of as a linear combination of columns.

Matrix multiplication

Multiplication on the left by a vector can be thought of a forming a linear combination of *rows.*

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 1 & 9 & -3 \\ -2 & 1 & 7 \end{bmatrix} =$$

Result is a row vector

$$a \begin{bmatrix} 2 & 4 & 5 \end{bmatrix} + b \begin{bmatrix} 1 & 9 & -3 \end{bmatrix} + c \begin{bmatrix} -2 & 1 & 7 \end{bmatrix}$$

Multiplication on the left can be thought of as a linear combination of rows.