Polynomial interpolation

The Lagrange Formula for polynomial interpolation

Vandermonde system

$$\underbrace{\begin{bmatrix} x_0^n & x_0^{n-1} & \dots & x_0 & 1 \\ x_1^n & x_1^{n-1} & \dots & x_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \dots & x_n & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\mathbf{y}}$$

We solved $V\mathbf{a} = \mathbf{y}$ to get the coefficients of the polynomial

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

Disadvantages to solving the Vandermonde system?

- Requires a linear solve (expensive)
- Potentially numerically ill-conditioned for large N.

Lagrange Polynomials

There are *explicit* (does not require a linear solve) ways of finding an interpolating polynomial through a given set of data points.

Given a set of data points (x_i, y_i), $i = 1, \ldots N+1$, suppose we had a set of polynomials $\ell_j(x)$ that satisfied

$$\ell_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(Such a set of polynomials exist; the coefficients $\mathbf{a}_j$ for the j^{th} polynomial $\ell_j(x)$ *could* be found by solving the Vandermonde system $V\mathbf{a}_j = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j^{th} column of the identity matrix. The $\mathbf{a}_j$ appear in the j^{th} column of V^{-1}.

Lagrange Polynomials

These polynomials are called the *Lagrange Interpolating Polynomials.*

$$\ell_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and allow us to explicitly write down the polynomial that interpolates the data

$$P_n(x) = \sum_{j=0}^{n} \ell_j(x) y_j$$

Check : $P_n(x_i) = y_i$ (by construction). The n^{th} degree interpolating polynomial through $n+1$ points is unique, so we must have the same polynomial as was found by solving Vandermonde system

Lagrange basis functions

The Lagrange basis functions can be easily computed :

Let

$$\ell_j(x) = a \prod_{k=0,\ k\neq j}^{n} (x - x_k)$$

We want $\ell_j(x_j) = 1$, so we set

$$a = \frac{1}{\prod_{k=0,\ k\neq j}^{n} (x_j - x_k)}$$

and we have an explicit formula for the interpolating polynomial.

Lagrange Formulation

The Lagrange form of the interpolating polynomial is given by

$$P_n(x) = \sum_{j=0}^{n} \ell_j(x) y_j$$

where

$$\ell_j(x) = \frac{\prod_{k=0,\ k\neq j}^{n} (x - x_k)}{\prod_{k=0,\ k\neq j}^{n} (x_j - x_k)}$$

Example - Fitting a quadratic

Find the parabola that fits through 3 data points :

$$(-1,2), \quad (0,3), \quad (2,-7)$$

$$\begin{aligned}
\ell_0(x) &= \frac{(x-0)(x-2)}{(-1-0)(-1-2)} &= \tfrac{1}{3}x^2 - \tfrac{2}{3}x \\
\ell_1(x) &= \frac{(x+1)(x-2)}{(0+1)(0-2)} &= \tfrac{1}{2}x^2 + \tfrac{1}{2}x + 1 \\
\ell_2(x) &= \frac{(x+1)(x-0)}{(2+1)(2-0)} &= \tfrac{1}{6}x^2 + \tfrac{1}{6}x
\end{aligned}$$

Check that $\ell_i(x_i) = 1$ and that $\ell_i(x_j) = 0$, $i \neq j$.

Check

The interpolating polynomial is then

$$P_2(x) = 2\ell_0(x) + 3\ell_1(x) - 7\ell_2(x) = -2x^2 - x + 3$$