The final is comprehensive, and will cover Chapters 1-5.

1. Write down the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

2. What is $(A+B)^2$ for square matrices A and B? What is $(AB)^2$?

3. Suppose A and B are both square, invertible matrices. Show that AB is invertible by at least two different methods.

4. Find the inverse of the following matrix. You may use any method we have learned in class.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

5. Solve the following system by first writing the system as a matrix system $A\mathbf{x} = \mathbf{b}$, then decomposing the matrix as $A = LU$, and finally solve using forward and back substitution.

$$\begin{aligned} 2u + 2v + 4w &= 0 \\ 8u + 9v + 19w &= 0 \\ 2u + 2v + 5w &= 1 \end{aligned}$$

6. Show that the vectors $\mathbf{x}$ that satisfy $A\mathbf{x} = 0$ form a subspace.

7. Find two vectors below that are perpendicular to the remaining two vectors.

$$\begin{bmatrix} 10 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -10 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

8. Find the nullspace of the following matrix.

$$A = \begin{bmatrix} 1 & -10 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This shows that the (row space)/(column space) of A is __________ to the (null space)/(left nullspace) of A.

9. Find the left nullspace of the following matrix.

$$A = \begin{bmatrix} 10 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that the (row space)/(column space) of A is _________ to the (null space)/(left nullspace) of A.

10. Do the vectors $(1,1,3)$, $(2,3,6)$ and $(1,4,3)$ form a basis for $\mathbb{R}^3$?

11. Find values of a and b so the system

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}$$

has

(a) No solution,

(b) Infinitely many solutions,

(c) Exactly one solution

12. Find the best fit line through the point $(-1,4)$, $(0,5)$ and $(1,9)$.

13. The matrix P projects vectors in $\mathbb{R}^4$ onto the subspace generated by the vectors $(1,1,1,1)$, $(0,1,3,4)$ and $(1,0,1,0)$. Show that P has as one of its eigenvalues $\lambda = 1$.

14. Show that the constant α that minimizes the function

$$f(\alpha) = (3-\alpha)^2 + (-1-\alpha)^2 + (4-\alpha)^2 + (5-\alpha)^2$$

is $\alpha = 11/4$. Use techniques you learned from linear algebra, not Calculus!

15. Show that if $\dim(N(A)) > 0$, then $A^T A$ is singular. **Hint:** Start with a vector $\mathbf{x}$ in $N(A)$.

16. An $m \times n$ matrix A has rank 3. Fill in the blanks and circle the correct answers from (choice one)/(choice two). Each piece of information provided applies to the same matrix, and so answers from later questions may require information provided in earlier questions.

(a) The matrix A has (at most)/(at least) 3 rows.

(b) The left null space of A has dimension 5. The (column space)/(row space) is a subspace of $\mathcal{R}$____ .

(c) The nullspace has dimension 1. The matrix $A^T A$ is (singular)/(nonsingular) and the (column space)/(row space) is in $\mathcal{R}$____ .

(d) All vectors in $\mathcal{R}$____ can be written as linear combination of _________ basis vectors from C(A) and _________ vectors from _________ .

(e) The matrix A is in $\mathcal{R}$____ $\times$ ____ . Fill in the correct dimensions of A.

17. What is the point on the plane $x + y - z = 0$ closest to $b = (2,1,0)$?

18. Provide at least three *different* ways to compute the determinant of a square matrix.

19. Alan and Barbara play a game in which they take turns filling in entries of an initially empty 4×4 matrix. Alan plays first. At each turn, a player chooses a real number of places it in a vacant entry. The game ends when all entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has the winning strategy?

    (a) What key property of a singular matrix could be used in this game?

20. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

21. An $n \times n$ matrix A has $k < n$ linearly independent eigenvectors. Show that this means that $k \leq \dim(C(A)) \leq n$. Try to find an example of a 3×3 matrix with full rank, but with only two linearly independent eigenvectors.

22. Consider all 4×4 matrices A that are diagonalized by the same fixed eigenvector matrix S. Show that the A's form a subspace. What is this subspace when $S = I$? What is its dimension?