26. Establish Eq. (24) to finish the argument for the extension of Green's theorem.

The Area Corollary of Green's Theorem

If a region $R$ and its simple closed curve boundary $C$ satisfy the hypotheses of Green's theorem, then the area of $R$ is given by the formula

$$\text{Area of } R = \frac{1}{2} \oint_C x \, dy - y \, dx.$$  \hspace{1cm} (29)

The reason is that by Eq. (16), run backward,

$$\text{Area of } R = \iint_R \frac{1}{2} x \, dy - \frac{1}{2} y \, dx.$$

Use Eq. (29) to calculate the areas in Problems 27–30.

27. The area of the circle

$$R(t) = (a \cos t)i + (a \sin t)j, \quad 0 \leq t \leq 2\pi$$

28. The area of the ellipse

$$R(t) = (a \cos t)i + (b \sin t)j, \quad 0 \leq t \leq 2\pi$$

29. The area enclosed by the hypocycloid

$$R(t) = (a \cos^3 t)i + (a \sin^3 t)j, \quad 0 \leq t \leq 2\pi$$

(See Fig. 5.35, page 334.)

30. The area enclosed by the curve

$$R(t) = t^3i + (t^3 - 3t)j, \quad -\sqrt{3} \leq t \leq \sqrt{3}$$

(See Fig. 5.36.)

### TOOLKIT PROGRAMS

Scalar Fields  Vector Fields
These programs evaluate line integrals.

## 19.4

### Surface Area and Surface Integrals

We know how to integrate a function over a flat region in the plane, but what if the function is defined over a curved surface instead? How do we calculate its integral then? The trick to evaluating one of these so-called surface integrals is to rewrite it as a double integral over a region in a coordinate plane beneath the surface (Fig. 19.30). This changes the surface integral into the kind of integral we already know how to evaluate.

Our first step is to find a double-integral formula for calculating the area of a curved surface. We will then see how some of the ideas that arise in the process can be used again to define and evaluate surface integrals. With surface integrals under control, we shall be able to calculate the flux of a three-dimensional vector field through a surface and the masses and moments of thin shells of material. In Articles 19.5 and 19.6 we shall see how surface integrals provide just what we need to generalize the two forms of Green's theorem to three dimensions. One of these generalizations will enable us to express the flux of a vector field through a closed surface as a triple integral over the three-dimensional region enclosed by the surface. The other generalization will enable us to express the circulation of a vector field around a closed curve in space as an integral over a surface bounded by the curve. These results have far-reaching consequences in mathematics as well as in the theories of electromagnetism and fluid flow.

### Surface Area

Figure 19.31 shows a piece $S$ of a surface $F(x, y, z) = c$ lying above its "shadow" region $R$ in a plane directly beneath it. If $F$ and its first partial derivatives are continuous, the area of $S$ can be defined and calculated as a
19.31 A surface $S$ and its vertical projection onto a plane beneath it. You can think of $R$ as the shadow of $S$ on the plane. The tangent plate $\Delta P_k$ approximates the surface patch $\Delta \sigma_k$ above $\Delta A_k$.

double integral over $R$. The origin of the integral takes a while to describe, but the integral itself is easy to work with.

The first step in defining the area of $S$ is to divide the region $R$ into small rectangles $\Delta A_k$ of the kind we would use if we were defining an integral over $R$. Directly above each $\Delta A_k$ lies a patch of surface $\Delta \sigma_k$ that we may approximate with a portion $\Delta P_k$ of the tangent plane. To be specific, we suppose that $\Delta P_k$ is a portion of the plane that is tangent to the surface at the point $(x_k, y_k, z_k)$ directly above the back corner $C_k$ of $\Delta A_k$. If the tangent plane is parallel to $R$, then $\Delta P_k$ will be congruent to $\Delta A_k$. Otherwise, it will be a parallelogram whose area is somewhat larger than the area of $\Delta A_k$.

Figure 19.32 gives a magnified view of $\Delta \sigma_k$ and $\Delta P_k$, showing the gradient vector $\nabla F(x_k, y_k, z_k)$ and a unit vector $\mathbf{p}$ that is normal to $R$. The figure also shows the angle $\gamma$ between $\nabla F$ and $\mathbf{p}$. The other vectors in the picture, $\mathbf{u}$ and $\mathbf{v}$, lie along the edges of the patch $\Delta P_k$ in the tangent plane. Thus, both $\mathbf{u} \times \mathbf{v}$ and $\nabla F$ are normal to the tangent plane.

We now need a fact we haven’t used since Article 13.7, namely that $|\mathbf{u} \times \mathbf{v} \cdot \mathbf{p}|$ is the area of the projection of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ onto any plane whose normal is $\mathbf{p}$. In our case, this translates into the statement

$$|\mathbf{u} \times \mathbf{v} \cdot \mathbf{p}| = \Delta A_k,$$  \hspace{1cm} (1)

Now, $|\mathbf{u} \times \mathbf{v}|$ itself is the area $\Delta P_k$ (standard fact about cross products) so Eq. (1) becomes

$$|\mathbf{u} \times \mathbf{v}| |\mathbf{p}| \cos \gamma = \Delta A_k,$$ \hspace{1cm} (2)

Same as $|\cos \gamma|$ because $\nabla F$ and $\mathbf{u} \times \mathbf{v}$ are both normal to the tangent plane

or

$$\Delta P_k |\cos \gamma| = \Delta A_k,$$  \hspace{1cm} (3)

or

$$\Delta P_k \frac{\Delta A_k}{|\cos \gamma|},$$

provided $\gamma \neq 0$. We will have $\cos \gamma \neq 0$ as long as $\nabla F$ is not parallel to the ground plane, or as long as $\nabla F \cdot \mathbf{p} \neq 0$.

Since the patches $\Delta P_k$ approximate the surface patches $\Delta \sigma_k$ that fit together to make $S$, the sum

$$\sum \Delta P_k = \sum \frac{\Delta A_k}{|\cos \gamma|},$$  \hspace{1cm} (4)

looks like an approximation of what we might like to call the surface area of $S$.

It also looks as if the approximation would improve if we refined the subdivision of $R$. In fact, the sums on the right-hand side of Eq. (4) are approximating
s for the double integral

\[ \int_R \frac{1}{|\cos \gamma|} \, dA. \]  

(4)

We therefore define the area of \( S \) to be the value of this integral whenever the integral exists.

For any particular surface \( F(x, y, z) = c \),

\[ |\nabla F \cdot \mathbf{p}| = |\nabla F| \, |\mathbf{p}| \, |\cos \gamma| = |\nabla F| \cos \gamma \]

and

\[ \frac{1}{|\cos \gamma|} = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}. \]

This combines with Eq. (4) to give a practical formula for calculating the area of the surface over a region \( R \).

\[ \text{Surface area} = \int_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA \]  

(5)

Thus, the area of the surface \( F(x, y, z) = c \) over a plane region \( R \) is the double integral over \( R \) of the magnitude of the gradient of \( F \) divided by the scalar component of the gradient normal to \( R \).

We reached Eq. (5) under the assumptions that \( \nabla F \cdot \mathbf{p} \neq 0 \) and that \( F \) and its first partial derivatives were continuous over \( R \) (so that \( \nabla F \) would be defined and continuous over \( R \)). Whenever the integral exists, however, we may define its value to be the area of the surface that lies over \( R \).

The surface area defined by Eq. (5) agrees with our earlier definitions of surface area. We shall not prove this, but see Problem 12.

**EXAMPLE 1** Find the area of the surface cut from the bottom of the paraboloid \( x^2 + y^2 - z = 0 \) by the plane \( z = 1 \).

**Solution** We sketch the surface and the region below it in the \( xy \)-plane (Fig. 19.33).

The surface \( S \) is part of the level surface \( F(x, y, z) = x^2 + y^2 - z = 0 \) and the region \( R \) below it is the disk \( x^2 + y^2 \leq 1 \). To get a unit vector normal to the plane of \( R \) we can take \( \mathbf{p} = \mathbf{k} \).

At any point \( (x, y, z) \) on the surface,

\[ F(x, y, z) = x^2 + y^2 - z, \]

\[ \nabla F = 2x \mathbf{i} + 2y \mathbf{j} - \mathbf{k}, \]

\[ |\nabla F| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1}, \]

and

\[ |\nabla F \cdot \mathbf{p}| = |\nabla F \cdot \mathbf{k}| = |-1| = 1. \]
The surface area is therefore

\[
S = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA = \iint_{x^2 + y^2 \leq 1} \sqrt{4x^2 + 4y^2 + 1} \, dx \, dy
\]

\[
= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} \, r \, dr \, d\theta \quad \text{ (polar coordinates)}
\]

\[
= \int_0^{2\pi} \left[ \frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^1 \, d\theta
\]

\[
= \int_0^{2\pi} \frac{1}{12} (5^{3/2} - 1) \, d\theta = \frac{\pi}{6} (5\sqrt{5} - 1).
\]

\[
\text{EXAMPLE 2} \quad \text{Find the area of the cap cut from the hemisphere } x^2 + y^2 + z^2 = 2, \ z \geq 0, \ \text{by the cylinder } x^2 + y^2 = 1 \ (\text{Fig. 19.34}).
\]

\textbf{Solution} \quad \text{The cap is part of the surface } F(x, y, z) = x^2 + y^2 + z^2 = 2. \ \text{It projects onto the disk } R: x^2 + y^2 \leq 1 \ \text{in the xy-plane. The vector } \mathbf{p} = \mathbf{k} \ \text{is normal to the plane of } R.

\text{At any point on the surface,}

\[
F(x, y, z) = x^2 + y^2 + z^2,
\]

\[
\nabla F = 2xi + 2yj + 2zk,
\]

\[
|\nabla F| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2},
\]

and

\[
|\nabla F \cdot \mathbf{p}| = |\nabla F \cdot \mathbf{k}| = 2z = 2z.
\]

\text{Therefore,}

\[
\text{Surface area } = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA = \iint_R \frac{2\sqrt{2}}{2z} \, dA = \sqrt{2} \iint_R \frac{dA}{z}.
\]

What do we do about the } z? 

Since } z \text{ is the } z\text{-coordinate of a point on the sphere, we may express } z \text{ in terms of } x \text{ and } y \text{ as}

\[
z = \sqrt{2 - x^2 - y^2}.
\]

We continue the work of Eq. (6) with this substitution:

\[
\text{Surface area } = \sqrt{2} \iint_R dA = \sqrt{2} \int_{x^2 + y^2 \leq 1} \frac{dA}{\sqrt{2 - x^2 - y^2}}
\]

\[
= \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r \, dr \, d\theta}{\sqrt{2 - r^2}} \quad \text{ (polar coordinates)}
\]

\[
= \sqrt{2} \int_0^{2\pi} \left[ -\frac{1}{2} (2 - r^2)^{1/2} \right]_r^1 \, d\theta
\]

\[
= \sqrt{2} \int_0^{2\pi} (\sqrt{2} - 1) \, d\theta
\]

\[
= 2\pi (2 - \sqrt{2}).
\]
Surface Integrals

We now show how to integrate a function over a surface, using the ideas we just examined for calculating surface area.

Suppose, for example, that we have an electrical charge distributed over a surface \( F(x, y, z) = e \) like the one shown in Fig. 19.31 and that the function \( g(x, y, z) \) gives the charge per unit area (charge density) at each point on \( S \). Then we may calculate the total charge on \( S \) as an integral in the following way.

We subdivide the shadow region \( R \) on the ground plane beneath the surface into small rectangles of the kind we would use if we were defining the surface area of \( S \). Then directly above each \( \Delta A_k \) lies a patch of surface \( \Delta \sigma_k \) that we approximate with a parallelogram-shaped portion of tangent plane, \( \Delta P_k \).

Up to this point the construction proceeds as in the definition of surface area, but now we take one additional step: We evaluate \( g \) at \((x_k, y_k, z_k)\) and approximate the total charge on the surface patch \( \Delta \sigma_k \) by the product

\[
g(x_k, y_k, z_k) \Delta P_k.
\]

The rationale is that when the subdivision of \( R \) is sufficiently fine, the value of \( g \) throughout \( \Delta \sigma_k \) is nearly constant and \( \Delta P_k \) is nearly the same as \( \Delta \sigma_k \). The total charge over \( S \) is then approximated by the sum

\[
\text{Total charge} = \sum g(x_k, y_k, z_k) \Delta P_k = \sum g(x_k, y_k, z_k) \frac{\Delta A_k}{\cos \gamma}.
\]

(7)

If \( F \), the function defining the surface \( S \), and its first partial derivatives are continuous, and if \( g \) is continuous over \( S \), then the terms on the right-hand side of Eq. (7) approach the limit

\[
\iiint_R g(x, y, z) \frac{dA}{\cos \gamma} = \iint_S g(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA
\]

as the rectangular subdivision of \( R \) is refined in the usual way. This limit is called the integral of \( g \) over the surface \( S \) and is calculated as a double integral over \( R \). The value of the integral is the total charge on the surface \( S \).

As you might expect, the formula in Eq. (8) defines the integral of any function \( g \) over the surface \( S \) as long as the integral exists. The integral's value takes on different meanings in different applications. If \( g \) is the constant function whose value is 1, the integral gives the surface area. If \( g(x, y, z) \) is the mass density of a thin shell of material modeled by a surface \( S \), the integral gives the mass of the shell.

We often abbreviate the integral in Eq. (8) as

\[
\iint_R g(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA = \iint_S g \, d\sigma,
\]

(9)

where \( d\sigma \) is the surface area differential

\[
d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA = \frac{dA}{\cos \gamma}.
\]

(10)

Surface integrals have all the usual algebraic properties of double integrals, the integral of the sum of two functions being the sum of their integrals and so
on. The domain additivity property takes the form
\[ \int \int_S g \, d\sigma = \int \int_{S_1} g \, d\sigma + \int \int_{S_2} g \, d\sigma + \cdots + \int \int_{S_n} g \, d\sigma. \]

The idea is that if \( S \) is partitioned by smooth curves and line segments into a finite number of nonoverlapping smooth patches, then the integral of a function \( g \) over \( S \) is the sum of the integrals over the patches. Thus, the integral of a function over the surface of a cube is the sum of the integrals over the faces of the cube. We integrate over a "turtle shell" of welded plates by integrating one plate at a time and adding the results.

**Example 3** Integrate \( g(x, y, z) = xyz \) over the cube cut from the first octant by the planes \( x = 1, y = 1, \) and \( z = 1. \)

**Solution** The cube is shown in Fig. 19.35. We integrate the product \( xyz \) over each face and add the results. Since \( xyz = 0 \) on the faces that lie in the coordinate planes, the integral of \( xyz \) over each of these faces is \( 0 \) and the integral over the cube reduces to
\[ \int \int \text{cube} \quad xyz \, d\sigma = \int \int \text{face } A \quad xyz \, d\sigma + \int \int \text{face } B \quad xyz \, d\sigma + \int \int \text{face } C \quad xyz \, d\sigma. \]

Face \( A \) is the surface \( F(x, y, z) = z = 1 \) over the square region \( R_{xy} \); \( 0 \leq x \leq 1, \) \( 0 \leq y \leq 1, \) in the \( xy \)-plane. For this surface and region,
\[ p = k, \quad \nabla F = k, \quad |\nabla F| = 1, \quad |\nabla F \cdot p| = 1, \]
and
\[ d\sigma = \frac{|\nabla F|}{|\nabla F \cdot p|} \, dx \, dy, \quad xyz = xy(1) = xy, \]

and
\[ \int \int_{R_{xy}} xyz \, d\sigma = \int_0^1 \int_0^1 xy \, dx \, dy = \int_0^1 \frac{y}{2} \, dy = \frac{1}{4}. \]

Symmetry tells us that the integrals of \( xyz \) over \( B \) and \( C \) are also \( 1/4. \) Hence,
\[ \int \int_{\text{cube}} xyz \, d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}. \]

The Surface Integral for Flux
If we can choose a unit normal vector \( \mathbf{n} \) on a surface \( S \) in such a way that \( \mathbf{n} \) varies continuously as its initial point moves about the surface, we call the surface orientable or two-sided. Spheres and other closed surfaces in space (surfaces that enclose solids) are orientable, and by convention we choose \( \mathbf{n} \) on a closed surface to point outward. Once \( \mathbf{n} \) is chosen we call the surface an oriented surface. The direction of \( \mathbf{n} \) at any point is called the positive direction. Any patch or subportion of an orientable surface is also orientable. The Möbius band shown in Fig. 19.50 in Article 19.6, which is one-sided, is nonorientable.
If \( \mathbf{F} \) is a continuous vector field defined over a surface \( S \), we call the integral over \( S \) of \( \mathbf{F} \cdot \mathbf{n} \), the scalar component of \( \mathbf{F} \) in the direction of \( \mathbf{n} \), the flux of \( \mathbf{F} \) across \( S \) in the positive direction.

**Definition**

The flux of \( \mathbf{F} \) across \( S \) in the direction of \( \mathbf{n} \) is

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.
\]  

(1)

This definition is a direct generalization of the definition introduced in Article 19.2 for the flux of a two-dimensional vector field \( \mathbf{F} \) across a plane curve \( C \). In the plane, the flux is

\[
\int_C \mathbf{F} \cdot \mathbf{n} \, ds,
\]

which is the integral of the scalar component of \( \mathbf{F} \) normal to the curve. In space, the flux of a three-dimensional field \( \mathbf{F} \) across a surface is the integral of the scalar component of \( \mathbf{F} \) normal to the surface.

If \( \mathbf{F} \) is the velocity field of a three-dimensional fluid flow, then the flux is the net rate at which the fluid is crossing \( S \) in the positive direction. We shall discuss three-dimensional fluid flows in more detail in Article 19.6.

If \( S \) is part of a level surface \( G(x, y, z) = c \), then \( \mathbf{n} \) may be taken to be one of the two vectors

\[
\mathbf{n} = + \frac{\nabla G}{|\nabla G|} \quad \text{or} \quad \mathbf{n} = - \frac{\nabla G}{|\nabla G|},
\]

depending on which vector gives the preferred direction.

**Example 4** Find the flux of \( \mathbf{F} = yz \mathbf{j} + z^2 \mathbf{k} \) outward through the surface \( S \) cut from the cylinder \( y^2 + z^2 = a^2 \), \( z \geq 0 \), by the planes \( x = 0 \) and \( x = a \) (Fig. 19.36).

**Solution** The outward unit normal to \( S \) may be calculated from the gradient of \( G(x, y, z) = y^2 + z^2 \) to be

\[
\mathbf{n} = + \frac{\nabla G}{|\nabla G|} = \frac{2yz \mathbf{j} + 2z \mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2yz \mathbf{j} + 2z \mathbf{k}}{2a} = \frac{y}{a} \mathbf{j} + \frac{z}{a} \mathbf{k}.
\]

We also have

\[
d\sigma = \frac{|\nabla G|}{|\nabla G \cdot \mathbf{k}|} \, dA = \frac{2}{2z} \, dA = \frac{a}{z} \, dA,
\]

where we drop the absolute value signs because \( z \geq 0 \) on \( S \).

On the surface \( S \), the value \( \mathbf{F} \cdot \mathbf{n} \) is given by the formula

\[
\mathbf{F} \cdot \mathbf{n} = (yz \mathbf{j} + z^2 \mathbf{k}) \cdot \left( \frac{y}{a} \mathbf{j} + \frac{z}{a} \mathbf{k} \right)
\]

\[
= \frac{y^2z + z^3}{a} = \frac{z}{a} (y^2 + z^2)
\]

\[
= \frac{z}{a} (a^2) \quad (y^2 + z^2 = a^2 \text{ on } S)
\]

\[
= az.
\]
Therefore, the flux of \( \mathbf{F} \) outward through \( S \) is
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S (ax) \frac{a}{z} \, dA = \iint_{R_{ax}} a^2 \, dx \, dy = a^2 \cdot \text{area}(R_{xy}) = 2a^4.
\]

Masses and Moments
In engineering and physics, thin shells of material like bowls, metal drums, and domes are modeled with surfaces. Their masses and moments are calculated with surface integrals, using the formulas listed in Table 19.2.

**EXAMPLE 5** Find the center of mass of a thin hemispherical shell of radius \( a \) and constant density \( \delta \).

**Solution** We model the shell with the hemisphere
\[
F(x, y, z) = x^2 + y^2 + z^2 = a^2, \quad z \geq 0.
\]
See Fig. 19.37. The center of mass then lies on the \( z \)-axis. Hence, \( \bar{x} = \bar{y} = 0 \).
It remains only to find \( \bar{z} \) from the formula
\[
\bar{z} = M_{yz}/M.
\]
The mass of the shell is
\[
M = \iint_S \delta \, d\sigma = \iint_S \delta \, d\sigma = 2\pi a^2 \delta,
\]
which is the surface area times the density.

<table>
<thead>
<tr>
<th>TABLE 19.2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mass and moment formulas for very thin shells</strong></td>
</tr>
</tbody>
</table>

Mass:
\[
M = \iint_S \delta(x, y, z) \, d\sigma \quad (\delta(x, y, z) = \text{density at } (x, y, z))
\]

First moments about the coordinate planes:
\[
M_{x\bar{y}} = \iint_S x \delta \, d\sigma, \quad M_{y\bar{z}} = \iint_S y \delta \, d\sigma, \quad M_{z\bar{x}} = \iint_S z \delta \, d\sigma
\]

Coordinates of center of mass:
\[
\bar{x} = M_{y\bar{z}}/M, \quad \bar{y} = M_{z\bar{x}}/M, \quad \bar{z} = M_{x\bar{y}}/M
\]

Moments of inertia:
\[
I_x = \iint_S (y^2 + z^2) \delta \, d\sigma, \quad I_y = \iint_S (x^2 + z^2) \delta \, d\sigma,
\]
\[
I_z = \iint_S (x^2 + y^2) \delta \, d\sigma, \quad I_L = \iint_S r^2 \delta \, d\sigma
\]

\( r(x, y, z) = \text{distance from point } (x, y, z) \text{ to line } L \)

Radius of gyration about a line \( L \):
\[
R_L = \sqrt{I_L/M}
\]
To evaluate the integral for $M_{xy}$, we take

$$\left| \nabla F \right| = |2xi + 2yj + 2zk| = 2\sqrt{x^2 + y^2 + z^2} = 2a,$$

$$\left| \nabla F \cdot \mathbf{p} \right| = \left| \nabla F \cdot \mathbf{k} \right| = |2z| = 2z,$$

$$d\sigma = \frac{\left| \nabla F \right|}{\left| \nabla F \cdot \mathbf{p} \right|} dA = \frac{a}{z} dA.$$

This gives

$$M_{xy} = \int \int_S z \delta d\sigma = \delta \int \int_R \frac{a}{z} dA = \delta \int \int_R dA = \delta \ a(\pi a^2) = \pi a^3 \delta.$$

The $z$-coordinate of the center of mass is therefore

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi a^3 \delta}{2\pi a^2 \delta} = \frac{a}{2}.$$

The center of mass is the point $(0, 0, a/2)$.

---

**PROBLEMS**

**Surface Area**

1. Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

2. Find the area of the band cut from the paraboloid $x^2 + y^2 - z = 0$ by the planes $z = 2$ and $z = 6$.

3. Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.

4. Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the $xy$-plane.

5. Find the area of the surface $x^2 - 2y - 2z = 0$ that lies above the triangle bounded by the lines $x = 2$, $y = 0$, and $y = 3x$ in the $xy$-plane.

6. Find the area of the surface $x^2 + 2\ln x + \sqrt{15} y - z = 0$ above the square $R: 0 \leq y \leq 1, 1 \leq x \leq 2$ in the $xy$-plane.

7. Find the area of the cap cut from the top of the sphere $x^2 + y^2 + z^2 = 1$ by the plane $z = \sqrt{2}/2$.

8. Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

9. Find the area of the ellipse cut from the plane $z = cx$ by the cylinder $x^2 + y^2 = 1$.

10. Find the area of the portion of the cylinder $x^2 + z^2 = 1$ that lies between the planes $x = \pm 1/2$ and $y = \pm 1/2$.

11. Find the area of the surface cut from the paraboloid $x^2 + y + z^2 = 1$ by the plane $y = 0$. (Hint: Project the surface onto the $xz$-plane.)

12. Whenever we replace an old definition with a new one it is a good idea to try out the new one on familiar objects to see that it gives the answers we expect. For instance, is the surface area of a circular cylinder of base radius $r$ and height $h$ still $2\pi rh$? Find out by using Eq. (5) with $p = \mathbf{k}$ to calculate the area of the surface cut from the cylinder $y^2 + z^2 = r^2$ by the planes $x = 0$ and $x = h > 0$. (Find the area above the $xy$-plane, and double.)

**Surface Integrals**

13. Integrate $g(x, y, z) = x + y + z$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

14. Integrate $g(x, y, z) = y + z$ over the surface of the wedge shown in Fig. 19.38.

19.38 The wedge in Problem 14.

15. Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid cut from the first octant by the planes $x = a$, $y = b$, and $z = c$. 
16. Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a$, $y = \pm b$, and $z = \pm c$. (Hint: You will see what’s going on if you integrate over a pair of opposite faces.)

17. Integrate $g(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

18. Integrate $g(x, y, z) = yz$ over the portion of the plane $x + y + z = 1$ that lies in the first octant.

19. Integrate $g(x, y, z) = x^2 + y^2 + z^2$ over the surface cut from the cylinder $y^2 + z^2 = a^2$ by the planes $x = 0$ and $x = h$. Take $h > 0$.

20. Integrate $g(x, y, z) = x^2 + y^2$ over the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

Flux across a Surface

In Problems 21–26, find the flux of the vector field $\mathbf{F}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

21. $\mathbf{F}(x, y, z) = \mathbf{z}\mathbf{k}$

22. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

23. $\mathbf{F}(x, y, z) = y\mathbf{j} - x\mathbf{i} + \mathbf{k}$

24. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$

26. $\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

27. Find the flux of the field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ upward through the surface cut from the parabolic cylinder $z = 4 - x^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.

28. Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$ upward through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

29. Let $S$ be the portion of the cylinder $y = e^z$ in the first octant that projects parallel to the $x$-axis onto the rectangle $R_{yz}$: $1 \leq x \leq 2$, $0 \leq y \leq 1$ in the $yz$-plane (Fig. 19.39). Let $n$ be the unit vector normal to $S$ that points away from the $xy$-plane. Find the flux of the field $\mathbf{F}(x, y, z) = -2i + 2yj + zk$ across $S$ in the direction of $n$.

30. Let $S$ be the portion of the cylinder $y = \ln x$ in the first octant whose projection parallel to the $y$-axis onto the $x$-plane is the rectangle $R_{xy}$: $1 \leq x \leq e$, $0 \leq z \leq 1$. Let $n$ be the unit vector normal to $S$ that points away from the $xy$-plane. Find the flux of $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ through $S$ in the direction of $n$.

31. The sphere $x^2 + y^2 + z^2 = 25$ is cut by the plane $z = 1$, the smaller portion cut off forming a solid that is bounded by a closed surface $S$ made up of a spherical cap $S_1$ and a flat disk $S_2$. Find

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, d\sigma + \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ and in each integral on the right $n$ is taken to be the outward-pointing normal to the surface.

32. Find the outward flux of the field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

Masses and Moments

33. Find the center of mass of the portion of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant if the sphere’s density is constant.

34. Find the center of mass of the surface cut from the upper half of the cylinder $y^2 + z^2 = a^2$ by the planes $x = 0$ and $x = a$ if the density is constant. (This is the surface in Example 4.)

35. Find the center of mass, moment of inertia, and radius of gyration about the $z$-axis of a thin shell of constant density cut from the cone $z = \sqrt{x^2 + y^2}$ by the plane $z = 1$ and $z = 2$.

36. Find the moment of inertia about the $z$-axis of a thin shell of constant density cut from the cone $z = \sqrt{x^2 + y^2}$ by the cylinder $x^2 + y^2 = 2x$. (Hint: The cylindrical coordinate equation for the cylinder is $r = 2 \cos \theta$.)

37. Find the mass of a thin spherical shell of radius $a$ if $\mathbf{a}$ the density at each point is the distance from the point to a fixed diameter of the sphere, $\mathbf{b}$ the density at each point is the square of the distance from the point to a fixed diameter of the sphere.

38. Find the moment of inertia about the $z$-axis of the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ if the density is constant.