On metric spaces with the Haver property which are Menger spaces

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A metric space \((X,d)\) has the **Haver property** [Haver, 1974] if for each sequence \(\epsilon_1, \epsilon_2, \ldots\) of positive numbers there are disjoint open collections \(\mathcal{V}_1, \mathcal{V}_2, \ldots\) with diameters of members of \(\mathcal{V}_i < \epsilon_i\) and \(\bigcup_{i=1}^{\infty} \mathcal{V}_i\) covering \(X\).

A space \(X\) has the **property C** or is a **C-space** [Addis, Gresham, 1978], if for each sequence \(\mathcal{U}_1, \mathcal{U}_2, \ldots\) of open covers of \(X\) there are disjoint open collections \(\mathcal{V}_1, \mathcal{V}_2, \ldots\) with \(\mathcal{V}_i\) refining \(\mathcal{U}_i\) and \(\bigcup_{i=1}^{\infty} \mathcal{V}_i\) covering \(X\).

Such \(\{\mathcal{V}_i\}_{i=1}^{\infty}\) is called a **C-refinement** of \(\{\mathcal{U}_i\}_{i=1}^{\infty}\).

A metrizable space \(X\) has the property \(C\) if and only if for any metric \(d\) on \(X\) generating the topology, \((X,d)\) has the Haver property.

Every countable-dimensional space (i.e. a union of countably many 0-dimensional spaces) is a \(C\)-space and every \(C\)-space is weakly infinite-dimensional. The Hilbert cube fails the property \(C\).
Roman Pol, 1982: There exists a metrizable separable $C$-space $X$ such that $X \times X$ is not a $C$-space.

E.P., 1986: There exists a metrizable separable $C$-space $X$ such that $X \times Y$ is not a $C$-space for some subspace $Y$ of the space of irrationals $P$ (under CH, $Y = P$).

Jan van Mill, Roman Pol, 2006: There exists a completely metrizable separable $C$-space $X$ such that $X \times X$ is not a $C$-space.

The product $X \times Y$ of a $C$-space $X$ and a $\sigma$-compact $C$-space $Y$ is a $C$-space.
If \((X, d), (Y, e)\) are metric spaces, the product 
\((X, d) \times (Y, e)\) is equipped with the metric associating 
to \((x_1, y_1), (x_2, y_2) \in X \times Y\) the distance 
\(\max(d(x_1, x_2), e(y_1, y_2))\).

Liljana Babinkostova, Top. & Appl. 154 (2007):
(1) If \((X, d)\) has the Haver property and \((Y, e)\) is 
countable-dimensional, then \((X, d) \times (Y, e)\) has
the Haver property.
(2) Question. Is it true that \((X, d) \times (X, d)\) has
the Haver property for every \((X, d)\) such that
\(X\) has the property \(C\)?

Example, E.P. and R.P, PAMS 2008?
There exists a separable complete metric space
\((X, d)\) with the property \(C\) such that the square
\((X, d) \times (X, d)\) does not have the Haver pro-
perty.
For \( \sigma \)-compact spaces \( X \), the Haver property of \((X,d)\) implies the property \( C \) of \( X \) and the Haver property of any product \((X,d) \times (Y,e)\) by a metric space \((Y,e)\) with the Haver property.

We say that \( X \) is a **Menger** (resp. **Hurewicz**) space, if for each sequence \( U_1, U_2, \ldots \) of open covers of \( X \) there are finite open collections \( F_i \subset U_i, \ i = 1, 2, \ldots \), such that each \( x \in X \) belongs to some (resp. all but finitely many) \( \cup F_i \).

**Theorem.** (L. Babinkostova, 2007) Let \((X, d)\) be a metric space with the Haver property. If \( X \) is a Hurewicz space then

(i) \( X \) has the property \( C \),

(ii) for any metric space \((Y, e)\) with the Haver property, the metric space \((X, d) \times (Y, e)\) has the Haver property.

L. Babinkostova asked if the Hurewicz property can be replaced here by the Menger property. We will show that, under Martin's Axiom \( \text{MA} \), this is not the case.
**Theorem 1.** (E.P. and R.P.) Assuming **MA**, there exists a separable metrizable space \( E \) such that
(a) \( E \) has the Menger property,
(b) \((E, d)\) has the Haver property for some metric \( d \) on \( E \) generating the topology,
(c) \( E \) fails the property \( C \).
Moreover, under **MA**, for each compact metrizable space \( Z \) which fails the property \( C \) there is \( E \subset Z \) with these properties.

**Theorem 2.** (E.P. and R.P.) Assuming **MA**, there exists a separable metric space \((M, d)\) having the Haver property such that
(a) \( M \) is a Menger space,
(b) \( M \) is a \( C \)-space,
(c) the metric square \((M, d) \times (M, d)\) fails the Haver property.
Moreover, one can construct \( M \) so that \( M \times M \) has the Menger property.
Theorem 2 follows from the following example: there exists a separable metric space $E^* = E_0 \cup E_1$, where $E_0, E_1$ have the property $C$ and the Menger property, but there exists a metric $d_i$ generating the topology on $E_i$, for $i = 0, 1$, such that 

$$(E_0 \cap E_1, d_0 \vee d_1)$$

fails the Haver property, where 

$$(d_0 \vee d_1)(x, y) = \max(d_0(x, y), d_1(x, y)).$$

Indeed, consider the free union $M = E_0 \oplus E_1$ with the metric $d$ such that $d \mid E_i = d_i$. Then $M$ has the property $C$ (hence $(M, d)$ has the Haver property), but the map $x \rightarrow (x, x)$ from $E_0 \cap E_1$ into $E_0 \times E_1$ is an isometric embedding of $(E_0 \cap E_2, d_0 \vee d_1)$ into $(M, d) \times (M, d)$, hence this square fails the Haver property.
Notaion: \( I = [0, 1], Q \) - rationals from \( I \), \( P = I \setminus Q \).

**Lemma.** Assuming **MA**, there exists a family \( \{C_\alpha : \alpha < 2^\omega\} \) of pairwise disjoint countable dense subsets of the irrationals \( P \) from \( I = [0, 1] \) such that \( B = \bigcup \{C_\alpha : \alpha < 2^\omega\} \) has the following property:

(i) \(| B \setminus G | < 2^\omega\) for every dense \( G_\delta \)-set \( G \) in \( I \).

**Proof.** By a theorem of Martin and Solovay, if **MA** holds, then the intersection of less than \( 2^\omega \) dense \( G_\delta \)-sets of reals \( \mathbb{R} \) contains a dense \( G_\delta \)-subset of \( \mathbb{R} \). Thus one can fix a transfinite sequence \( G_1 \supset G_2 \supset \ldots \supset G_\xi \supset \ldots, \xi < 2^\omega \), of dense \( G_\delta \)-subsets of \( P \) such that any dense \( G_\delta \)-set in \( P \) contains some \( G_\xi \).

Now it suffices to choose, by a transfinite induction, pairwise disjoint countable dense sets \( C_\xi \subset G_\xi \).
Construction of the space $E$ from Theorem 1.

Let $X = \prod_{i=0}^{\infty} I_i$, where $I_i = [0,1]$, and let $p : \prod_{i=0}^{\infty} I_i \to I_0$ be the projection onto $I = I_0$.

Let $\{H_\xi : \xi < 2^\omega\}$, be the family of all $G_\delta$-sets in $X$ with $p(H_\xi) \supset B$.

We take a Hilgers function $h : B \to X$:
assuming $c \in C_\xi$,
if $p^{-1}(c) \setminus H_\xi \neq \emptyset$, we choose $h(c)$ from this set,
and we pick $h(c) \in p^{-1}(c)$ arbitrarily, whenever $p^{-1}(c) \subset H_\xi$.
We set $E = h(B)$ be the subspace of $X$.
It will be shown that $E$ is a Menger space without property $C$, such that for some metric $d$ generating the topology of $E$, $(E,d)$ is a Haver space.
We will use the following two basic properties of the set $E \subset X$:

**(I)** If $G$ is any $G_{\delta}$-set in $X$ containing $E$, then $G = H_{\xi}$ for some $\xi < 2^\omega$ and hence $G \supset p^{-1}(C_{\xi}) = \bigcup \{p^{-1}(c) : c \in C_{\xi}\}$, where every fiber $p^{-1}(c) = \{c\} \times \prod_{i=1}^{\infty} I_i$ is a copy of the Hilbert cube.

**(II)** If $U$ is an open set in $X$ containing some set $p^{-1}(C)$, where $C$ is dense in $I$, then $|E \setminus U| < 2^\omega$.

Indeed, $V = I \setminus p(X \setminus U)$ is open and dense in $I$ and $p^{-1}(V) \subset U$, hence by (i) $|B \setminus V| < 2^\omega$ and thus $|E \setminus U| < 2^\omega$. 
The space $E$ has the Menger property.

Let $G_1, G_2, \ldots$ be a sequence of open collections in $X$ covering $E$ and let $H = \bigcap_{j=1}^{\infty} (\bigcup G_{2j})$. Then $H \supset E$, hence by (I), there is $\xi < 2^\omega$ such that $p^{-1}(C_\xi) \subset H_\xi = H$.

Since $p^{-1}(C_\xi)$ is $\sigma$-compact, there are finite collection $F_{2j} \subset G_{2j}$ with $p^{-1}(C_\xi) \subset \bigcup_{i=1}^{\infty} \bigcup F_{2j} = U$.

By (II), $|E \setminus U| < 2^\omega$, hence $U \setminus E$ is a Menger space by a Fremlin-Miller theorem.

Thus there are finite collection $F_{2j+1} \subset G_{2j+1}$ with $(E \setminus U) \subset \bigcup_{i=1}^{\infty} \bigcup F_{2j+1}$.
The space $E$ fails the property $C$.

Let $C_m = \{(x_0, x_1, \ldots) \in X : x_m = 0\}$, $D_m = \{(x_0, x_1, \ldots) \in X : x_m = 1\}$, and let $U_m = \{X \setminus C_m, X \setminus D_m\}$, for $m = 1, 2, \ldots$.

For every $t \in I$, $\{C_m \cap p^{-1}(t), D_m \cap p^{-1}(t)\}_{i=1}^{\infty}$ is a sequence of pairs of opposite faces in the Hilbert cube $p^{-1}(t) = \{t\} \times \prod_{i=1}^{\infty} I_i$, hence the sequence $\{U_m \mid p^{-1}(t)\}_{i=1}^{\infty}$ does not have a $C$-refinement in $p^{-1}(t)$.

The sequence $\{U_m \mid E\}_{m=1}^{\infty}$ does not have any $C$-refinement in $E$.

Indeed, suppose that there exist disjoint collections $\mathcal{V}_m$, $m = 1, 2, \ldots$, of open sets in $X$ such that $\mathcal{V}_m \mid E$ refines $U_m$ and $V = \bigcup_{m=1}^{\infty} \bigcup \mathcal{V}_m \supset E$.

Then, by (I), $V \supset p^{-1}(C_\xi)$ for some $\xi$, hence for any $t \in C_\xi$, $V \supset p^{-1}(t)$.

It follows that $\{\mathcal{V}_m \mid p^{-1}(t)\}_{i=1}^{\infty}$ is a $C$-refinement of $\{U_m \mid p^{-1}(t)\}_{i=1}^{\infty}$ - a contradiction.
Let $p_n : \prod_{i=0}^{\infty} I_i \to \prod_{i=0}^{n} I_i$ be the projection for $n = 0, 1, \ldots$; hence $p = p_0$.

Arrange all the rationals from $Q$ into a sequence $\{q_1, q_2, \ldots\}$.

Let $T$ be the compactum obtained from $X$ by attaching to each set $p^{-1}(q_n) = \{q_n\} \times \prod_{i=1}^{\infty} I_i$ the cube $\{q_n\} \times \prod_{i=1}^{n} I_i$ by the mapping $p_n | p^{-1}(q_n)$.

Let $\pi : I^\mathbb{N} \to T$ be the quotient mapping, and let $T_n = \pi(p^{-1}(q_n))$ be a copy of $n$-dimensional cube.

Observe that $\pi | E : E \to \pi(E')$ is a homeomorphism.
Let \( \mathbb{N} = N_0 \cup N_1 \), where \( N_0 \cap N_1 = \emptyset \), 
\( \{ q_n : n \in N_s \} = I \), \( s = 0, 1 \).

Put \( S_s = \bigcup \{ T_j : j \in N_s \} \) and \( E_s = \pi(E) \cup S_s \subset T \), for \( s = 0, 1 \), and \( M = E_0 \oplus E_1 \).

The space \( E_s \) has the property \( C \) for \( s = 0, 1 \).
Since \( S_s \) is a countable union of finite-dimensional cubes, it has the property \( C \) and for every open set \( U \) in \( T \) containing \( S_s \), \( | E_s \setminus U | < 2^\omega \), hence is \( 0 \)-dimensional.
Thus \( E_s \) has the property \( C \) by the following simple fact: If \( N \subset M \subset T \) are such that \( N \) has the property \( C \), and for every open set \( U \) in \( T \) containing \( N \), \( M \setminus U \) has the property \( C \) then \( M \) has the property \( C \).

The space \( E_s \) has the Menger property, \( s = 0, 1 \).
Indeed, the space \( E_s \) is the union of \( \pi(E) \), which is homeomorphic to a Menger space \( E \) and \( \sigma \)-compact space \( S_s \), hence it is a Menger space.

There is a metric \( d \) on \( \pi(E) \) such that \( (E, d) \) has the Haver property.
Indeed, for every metric \( d \) generating the topology on \( E_0 \), \( (\pi(E), d \mid \pi(E)) \) has the Haver property.
There exist metrics $d_s$ on $E_s$, such that $(\pi(E), d_0 \vee d_1)$ does not have the Haver property, where $(d_0 \vee d_1)(x, y) = \max(d_0(x, y), d_1(x, y))$.

Let $A_m = \pi(C_m)$, $B_m = \pi(D_m)$, $F_s(m) = \bigcup\{T_j : j \in N_s, j \leq m\} \subset E_s$. Since, for $n \geq m$, $p_n(C_m) \cap p_n(D_m) = \emptyset$, we have $A_m \cap B_m \subset F_0(m) \cup F_1(m)$.

Fix open sets $V_0(m), V_1(m)$ in $T$ such that $F_s(m) \subset V_s(m)$, $V_0(m) \cap V_1(m) = \emptyset$. It follows that $A_m \cap V_{1-s}(m) \cap E_s$ and $B_m \cap V_{1-s}(m) \cap E_s$ have disjoint closures in $E_s$. Therefore, there are continuous maps $\varphi^s_m : E_s \to [0, 1]$ such that $\varphi^s_m|_{A_m \cap V_{1-s}(m)} \equiv 0$, $\varphi^s_m|_{B_m \cap V_{1-s}(m)} \equiv 1$.

Let $\rho$ be any metric on $T$ generating the topology. We set, for $s = 0, 1$, $d_s(x, y) = \rho(x, y) + \sum_{m=1}^{\infty} 2^{-m} | \varphi^s_m(x) - \varphi^s_m(y) |$, where $x, y \in E_s$. Then $d_s$ is a metric generating the topology on $E_s$. 
Remark. Under MA, there exists a metizable separable space $Y$ such that
(a) $Y$ has the Menger property,
(b) $Y$ has the property $C$,
(c) the product $Y \times P$ of $Y$ with the space of irrationals $P$ is strongly infinite-dimensional, and hence is not a $C$-space.
Moreover, one can construct $Y$ so that the square $Y^2$ has the Menger property and the property $C$.

Indeed, one can take $Y = E_0$.
Consider a mapping $f : E_0 : E_0 \to I_0$ be such that $p = f \circ \pi$, then $f^{-1}(P) = \pi(E)$ and the set $\text{Graph}(f | \pi(E)) = \{(x, f(x)) : x \in \pi(E)\} = F$ is homeomorphic to $\pi(E)$. It follows that the product $E_0 \times P$ is strongly infinite-dimensional, since it contains the strongly infinite-dimensional space $F$ as a closed subset.