Defining a Rational Function

A rational function is a function of the form \( \frac{f(x)}{g(x)} \), where \( f(x) \) and \( g(x) \) are polynomials and \( g(x) \neq 0 \). Recall that a polynomial is an expression made with constants, variables and exponents, which are combined using addition, subtraction and multiplication, but not division. (The variables are not an exponent. Those will be exponential expressions which we will discuss later in the semester.)

Looking more closely at a Rational Function

Young’s Rule:
Young’s Rule is a rule that applies to the dosage of a drug that can be given to a child if you know the adult dosage. The function \( D(C) = \frac{AC}{C+12} \), where \( D \) = dosage for a child, \( A \) = dosage for an adult, and \( C \) = child’s age in years. While we look at the application, we will also look at the characteristics of the rational function itself. First we need to make the adult dosage a constant. Let’s let the adult dosage of a certain drug be 325 milligrams so we are left with only 2 variables. Rewrite the function with this information. Explain why this represents a rational function.

After you have rewritten the function, graph the function on your calculator. Sketch the graph below. What is a good window for your graph? How did you choose this window?
**Domains:**
Sometimes it is hard to know if the window that we are using is large enough to see the entire graph. When making the window keep in mind the domain of the function. What is the domain of the function? Discuss the domain of the function both out of context and in context.

Look at your graph, what happens on your *graph* when $C = -12$? You might have to adjust the window on your calculator to really see this.

You should notice that your graph makes a “jump” at this point. Explore why there is the “jump” in the graph. What causes this jump? How do you know this?

We usually sketch asymptotes with a dashed line so make the line $C = -12$ a dashed line. When you sketch this line, make sure that your graph does not cross or touch this line. Your graph should get closer and closer to this line but never touch or cross it.

**Asymptotes:**
Since we just found an asymptote to our graph, let’s look more in depth at asymptotes. An asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. In other words, an asymptote is a line (or a curve) that a graph gets infinitely close to, but the graph never touches this line (or curve). (In our function, the line $C = -12$ is an asymptote that we have found. We will find another.)

**Vertical Asymptotes:**
The above asymptote that we found is called a vertical asymptote. How do you think you find the vertical asymptotes of a rational function? Why is this a vertical asymptote? What does it mean in our situation?
To find the vertical asymptote of a rational function we set the denominator equal to zero. We always write the vertical asymptote as $x = a$, or in our case $C = b$.

Find the vertical asymptote(s) for the following functions, explain how you found the asymptote:

1. $C(t) = \frac{4}{(t+1)(t-3)}$

2. $R(x) = \frac{x^2 + 1}{x^2 - 3x + 2}$

3. $P(m) = \frac{2m - 3}{m^2 + 1}$

**Horizontal/Oblique Asymptotes:**
A rational function either has a horizontal asymptote or a slant/oblique asymptote. Unlike a vertical asymptote, a rational function will only have one horizontal or slant/oblique asymptote.

Looking at the graph on your calculator, does the graph of our function have a horizontal asymptote? If so, what is it? How can you verify that you have found the correct horizontal asymptote? (Remember that your horizontal asymptote is written in the form $y = d$, or in our case $D(C) = e$.)

Sketch the asymptote on your sketched graph above. Remember to sketch it as a dashed line. Discuss what this asymptote means in the context of the problem. Why is the asymptotic behavior happening? How do you know this?

What would the dose be for a 6-year old? How do you know this?
What would the dose be for an 18-year old? How did you determine this?

What would the dose be for a 6 month old?

At what age will a child receive the same dose as an adult? Explain how you found this.

Let’s change our function to analyze the horizontal asymptote. Graph \( D(C) = \frac{225C}{C+12} \) on the same set of axis in your calculator. Does it have the same horizontal asymptote as our original function? Explain. (You might want to graph this function at the same time that you graph the original function just to see the difference.)

Discuss what you know about this function \( D(C) = \frac{225C}{C+12} \) (where the asymptotes are, what the asymptotes mean, what the adult dosage is etc.).

Without graphing, what do you think the horizontal asymptote of the function \( D(C) = \frac{125C}{C+12} \) will be? How do you know this? What does this asymptote mean in context?

**Range:**
The range of a function is the value(s) that \( D \) can be. Looking at the graph of the rational function, \( D(C) = \frac{325C}{C+12} \) above, what is the range of the function? What does the range mean in our context?

Summarize how to find the range in a rational function.
A Rational Function Who’s Graph Has a Hole:
Now we will look at a special case of the graph of a rational function. Let’s use the function
\[ R(t) = \frac{t + 1}{t^2 + 6t + 5}. \]
Before we graph this, what is/are the vertical asymptotes?

What is the horizontal asymptote if it exists?

What is the domain of the function?

What is the range of the function?

Before graphing the function on your calculator, sketch what you think the graph will look like on
the grid below.

Graph this function in your calculator. How does your sketched graph compare to your calculator
graph?

Now, simplify the rational function \( R(t) = \frac{t + 1}{t^2 + 6t + 5}. \)
Graph both the original function and the simplified function on the same graph. Explain the difference between the two graphs.

**Some extension questions about rational functions:**

**Comparing two rational functions:**

What are the similarities and differences between the graphs of the two following functions? $C(m) = \frac{2m-1}{m}$ and $C(m) = 2 - \frac{1}{m}$? Examine the domains of each, the range of each, and the asymptotes of each.

**Write the equation given specific examples:**

1. Write the equation of a rational function that has a vertical asymptote at $x = -2$. Write another equation that fits the guidelines. Discuss the similarities and differences between the two equations that you wrote.

2. Write the equation of a rational function that has a horizontal asymptote at $y = \frac{4}{5}$ and a vertical asymptote at $x = 3$ and $x = 0$. Share your equations with your group. Write another equation that fits the guidelines. Discuss the similarities and differences between the two equations that you wrote.

**Connections:**

Relating previous skills to a new skill; we plan to provide at least one problem on each test which expands previously learned skills to a new application. Today we will expand solving polynomial equations algebraically to solving them graphically.

You know how to solve the equation $x^2 + 2x - 15 = 0$ algebraically.

You know how to graph the function $y = x^2 + 2x - 15$. 
Begin by solving the equation \( x^2 + 2x - 15 = 0 \) algebraically by factoring:

Next graph the quadratic equation \( y = x^2 + 2x - 15 \) and sketch the graph below.

Discuss the connection between the graph of \( y = x^2 + 2x - 15 \), the solution to the equation \( x^2 + 2x - 15 = 0 \), and the factored form of \( x^2 + 2x - 15 \).

Now, graph the equation \( y = 2x^2 + 7x + 3 \). Using what you found from the previous problem, what is/are the solution(s) to the equation \( 2x^2 + 7x + 3 = 0 \)?

Let's expand on this by looking at a higher degree polynomial. We want to solve the equation \( x^4 - 15x^2 + 10x + 24 = 0 \). Begin by graphing the equation \( y = x^4 - 15x^2 + 10x + 24 \).

What are the solutions to the equation \( x^4 - 15x^2 + 10x + 24 = 0 \)?

What is the factored form of \( y = x^4 - 15x^2 + 10x + 24 \)?

How can you verify that you are correct?
Solve the equations below.

1. \( x^3 - 2x^2 - 6x + 8 = 0 \)
2. \( x^4 - 26x^2 + 25 = 0 \)
3. \( x^3 - 4x = -3x^2 + 12 \)

For #3, explain two different ways that you could solve that equation.