Mathematical modeling is simply the act of building a model (usually in the form of graphs) which provides a “picture” of a numerical situation.

Linear Modeling is mathematical modeling where the graph is a line, hence linear. Linear modeling occurs any time the rate of change is constant.

Before we begin with an example of a linear model, let’s begin by reexamining what a graph is. Last week, we discussed the coordinate system and talked about graphing points in different quadrants. But what is a point. Take some time to write down what a point is. (Consider graphing the equation \( y = 4x + 3 \). The first step is to find two points, the second step plot the points and lastly draw the line. What do these two points mean?)

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Next consider that we also use function notation for a line. We can rewrite the equation \( y = 4x + 3 \) using function notation as \( f(x) = 4x + 3 \). Now the point would be represented as \( (x, f(x)) \). In this equation, we often say that \( x \) is the input value and \( f(x) \) is the output value or answer. Consider the following problem:

A restaurant is promoting its famous chili with a special offer. If you eat dinner at the restaurant on Wednesday night, you will be given three free cans of chili. If you would like to purchase additional cans of chili, you may at a price of four for a dollar. Let \( x \), represent the amount of money you plan to spend on chili (in dollars) and write an equation to represent the amount.

Checking your answer, your function should be identical to the function we used as an example in the above paragraph.

Based on the chili problem, what does a point represent:

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Now that you have redefined a point in the chili problem, look over your answer to what a point is. Can you write a better answer? In your improved answer, please use the word relation or relationship.

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The equation developed is actually an algebra model relating the number of dollars spent to the number of cans of chili taken home. If we graph the equation, the result will be a picture which allows us to quickly and visually determine how much we need to spend to buy the number of desired cans of chili.

A mathematical model is designed to relate two or more conditions (numbers) so that projections can be made. The chili model is a very simple model. A complicated mathematical model would be the models used in predicting events such as weather. In schools, mathematical models are used to model a) enrollments to assist in predicting number of classes at each grade level for next year, b) test scores to assist in evaluating curriculum changes both from the perspective of new implementations and from the perspective of determining if change is necessary and c) budgets. However, these models are often not linear.

As educators, you will not only need to understand math to better understand the above information, but you will need to be prepared to insure that your students understand these concepts as it relates to their chosen field of future study. Thus, let’s look at the business example being used this week.

Business Example 1:

An automobile manufacturer has a factory which is open 24 hours a day and produces 10 finished cars per hour. This model has a constant rate of change in that each hour 10 additional cars are produced. Hence we will want to build a linear model. To begin to build our linear model, let’s pick two points

How many cars will be built in an hour? ______
Enter the answer in point form: (____, ____)

How many cars will be built in 24 hours? ______
Enter the answer in point form: (____, ____)

Now that we have two points, let’s create a graph. As we start graphing our two points, we draw and label two axes. But, what should we use as labels? What letters should we use to represent our variables? Since we currently are in a math course, it would be easy to choose x and y as our representative letters. But what do x and y mean? Using the letters, x and y, in math makes perfect sense as we are able to move from one mathematical concept to the next with a strong level of consistency and thus, math is easier to learn from chapter to chapter and course to course.
When we leave math and move to other subjects, such as business, we will choose letters which reflect the situation we are modeling. In this case, let’s use $t$ to represent time in hours and $n$ to represent the number of cars completed. In this case, a general point will be in the form $(t, n)$.

The variable $t$, representing time, is called the independent variable and will be placed on the horizontal axis when graphing.

The variable $n$, representing number of autos completed, is called the dependent variable (this is because the number of autos completed depends on the time allowed). The vertical axis will be used to represent $n$.

Having two points and knowing that this problem is represented by a linear model, the graph can be created. Use the set of axes below to draw the graph. Please label the $t$ and $n$ axes respectively. Be careful to think about the scale of the problem before beginning to place the points.

Now that the graph has been created, the graph can be used to estimate how many cars will be created at any given time. If we look at the graph and consider the scale, we recognize that the best the graph can do is a rough estimation. If the model is needed to have more accurate estimations, then we need to create a numerical solution for the graph. Thus, we need to create the equation of the line. This will require finding both the slope, $m$, and the $y$-intercept, $b$.

*Side note: Many students have difficulty when we use letters other than $x$ and $y$. When students enroll in other courses which use mathematical modeling and do not use $x$ and $y$ as the variable, they often have difficulty relating their other course to mathematics. In this case the new material is more difficult to learn.*

*Hence, the sooner the student recognizes that an equation using letters is math regardless of the letters being used, the easier their overall education will be.*
Finding the slope: To find the slope we use the equation

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

One might recognize that in this problem we no longer have an \( x \) or a \( y \) and be confused. But this is easy to overcome when we realize that all we did was replace \( x \) with \( t \) and \( y \) with \( n \), giving the equation as

\[ \text{slope} = \frac{n_2 - n_1}{t_2 - t_1} \]

Compute the slope.

Next we need to find the \( y \)-intercept. Recall that we changed the variable of \( y \) to \( n \). Hence we need to find the \( n \)-intercept. The actual intercept that we are looking for to complete the equation of the line is the intercept of the vertical axis. There are mathematical processes which can be used to find this value. However, in this case a little bit of mathematical reasoning will also find the value. The intercept occurs when \( t = 0 \). One can recognize when time is zero, there has not been any opportunity to complete a car and thus, \( n = 0 \). Hence, our \( n \)-intercept is 0. And we have the equation

\[ n = mt + 0 \]
\[ = mt \]

Rewrite the equation replacing the slope, \( m \), with the value you found for the slope above.

The equation is ______________________________

Finding the implied domain: There is one final step to completing this example and this step is to find the implied domain. Implied domain is the correct mathematical term but outside of math terms such as feasible region will be used.

Recall that domain represents the set of all possible \( x \)-values (sometimes referred to as input values). In the case of our example, the domain is the set of all \( t \)-values. Based on the equation written above, there is no restriction on the domain so the domain is all real numbers.

This problem has more to it than just an equation. It has a given set of conditions which allowed us to write an equation. What restrictions are put on the problems based on the conditions of completing automobiles? This restriction is called the implied domain (or feasible region).
On the lower side, one can easily find a boundary where the problem no longer makes sense. If we allow time to be minus one hour (-1), we find that the equation claims that we make negative ten (-10) automobiles. This does not make sense. Thus the feasible region (implied domain) only makes sense if we begin at $t = 0$. Thus our first step in finding a feasible region (implied domain) provides the set: $\{t: t \geq 0\}$.

Now for the right hand side: is it reasonable to expect $1,000,000,000,000,000$ automobiles to be created? On the other hand, is it feasible to produce $1,000,000,000,000,000$ automobiles without affecting the problem? What is the upper bound? On the right hand side, the feasible region is not limited by a single number and different groups will produce different values for the upper limit. In your group decide on an upper limit. Write the feasible region (implied domain) and draw a new graph.

End of Business Example 1.

Problems related to lines:

Find the equation of the line and graph given the following points.

1) $(0, 3)$ and $(5, 7)$
2) $(0, 6)$ and $(20, 26)$
3) $(0, -10)$ and $(5, 25)$
4) $(0, 15)$ and $(5, 7)$
The first four problems allowed for the y-intercept to be found based on the first point. Here is some more practice but the y-intercept is not given directly.

5) \((6, 3)\) and \((5, 7)\)
6) \((8, 13)\) and \((10, 7)\)
7) \((-5, 9)\) and \((4, 8)\)
8) \((-2, -4)\) and \((-8, -7)\)