Graphing Radicals
Business 7

Radical functions have the form:
\[ \sqrt{x}, \sqrt[3]{x}, \sqrt[4]{x} \text{ and so on.} \]

The most frequently used radical is the square root; since it is the most frequently used we assume the number 2 is used and the square root is written as:
\[ \sqrt{x} \]

Square roots are simplified by looking for a number which multiplies by itself to obtain the number inside the square root sign. For example
\[ \sqrt{1} = 1, \quad \sqrt{4} = 2, \quad \sqrt{9} = 3, \quad \sqrt{25} = 5 \text{ and so on.} \]

The superscript number in front of the radical, called the index, tells how many times a number must be multiplied. For example:
- cube root: \[ \sqrt[3]{27} = 3 \text{ since } 3 \times 3 \times 3 = 27 \]
- fourth root: \[ \sqrt[4]{16} = 2 \text{ since } 2 \times 2 \times 2 \times 2 = 16 \]
- fifth root: \[ \sqrt[5]{3125} = 5 \text{ since } 5 \times 5 \times 5 \times 5 \times 5 = 3125 \]

Solve the following problems.
1) \[ \sqrt{64} = \]
2) \[ \sqrt[3]{64} = \]
3) \[ \sqrt[5]{32} = \]

So far we have only provided problems that have integer answers. The reason for this is that when simplifying radicals, if the results are not integers, then the radical is an irrational number. In these cases the answer is best obtained through the use of a calculator. If a calculator is used, then the result will be approximate.

\[ \sqrt{5} = 2.236068 \ldots \]

In this case, since the \( \sqrt{5} \) is the exact answer, we typically will not rewrite the number as a decimal number and we will say that the \( \sqrt{5} \) is in simplified form.

Simplifying radicals when variables are inside the radical. First recall that exponents are indications of repeated multiplication so that
\[ y^7 = y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \]

Irrational numbers by definition are decimal numbers which do not have an end to the number of digits after the decimal point and the pattern of the digits never repeats. To signify an irrational number, we generally place three dots at the end to indicate that the number continues. Thus, \( 3.1214 \ldots \) is an irrational number.
Thus, \( \sqrt{y^7} = \sqrt{y \cdot y \cdot y \cdot y \cdot y \cdot y} \). Next, recall from simplifying numbers, we can group factors inside the square root sign into groups of twos and bring one outside for every two inside so that

\[
\sqrt{y^7} = \sqrt{(y \cdot y) \cdot (y \cdot y) \cdot (y \cdot y) \cdot y} = y^3 \sqrt{y}
\]

If we have a cube root, \( \sqrt[3]{\cdot} \), or a higher power root, then we just group the similar factors inside by the power of the root so that:

\[
\sqrt[3]{y^7} = \frac{3}{3}(y \cdot y \cdot y) \cdot (y \cdot y \cdot y) \cdot y = y^2 \sqrt[3]{y}
\]

\[
\sqrt[4]{y^7} = \frac{4}{4}(y \cdot y \cdot y \cdot y) \cdot y \cdot y = y^2 \sqrt[4]{y^3}
\]

\[
\sqrt[5]{y^7} = \frac{5}{5}(y \cdot y \cdot y \cdot y \cdot y) \cdot y \cdot y = y^2 \sqrt[5]{y^2}
\]

If more than one variable is in the equation, then we work with each variable in a similar fashion to obtain the same result. So that

\[
\sqrt{12x^5y^3} = \sqrt{(2 \cdot 2) \cdot 3 \cdot (x \cdot x) \cdot (x \cdot x) \cdot x \cdot (y \cdot y) \cdot y}
\]

\[
= 2x^2y\sqrt{3xy}
\]

Simplify the following radicals.

4) \( \sqrt{27x^3y^4} \)

5) \( \sqrt{64xy^4z^7} \)

6) \( \sqrt[3]{27x^3y^4} \)

7) \( \sqrt[4]{27x^3y^4} \)

8) \( \sqrt[5]{27x^3y^4} \)
Rewriting radicals with rational exponents: Radicals represent fractional exponents so we can rewrite radicals as follows:

\[ \sqrt[2]{?} = (?,)^{\frac{1}{2}} \]
\[ \sqrt[3]{?} = (?,)^{\frac{1}{3}} \]
\[ \sqrt[4]{?} = (?,)^{\frac{1}{4}} \]

Thus:

\[ \sqrt[3]{4x^2yz} = (4x^2yz)^{\frac{1}{3}} = 4^{1/3}x^{2/3}y^{1/3}z^{1/3} \]

Rewrite the following using fractional exponents.

10) \( \sqrt{27x^3y^4} \)

11) \( \sqrt[3]{64xy^4z^7} \)

12) \( \sqrt{3xyz} \)

13) \( \sqrt[4]{3xy^2x^3} \)

Graphing Radical Functions

The domain of a square root function is not all real numbers. Consider \( \sqrt{-9} \), if this had a solution, we would need two numbers that when multiplied together create a negative number. This is not possible in the set of real numbers. Thus we say that the solution to \( \sqrt{-9} \) does not exist in the real number system. Hence, square root functions are one of the first type of functions which result in a restricted domain based on the equation. Can you name any others?
Next consider $\sqrt{0}$, does this have a solution in the real number system? At this level of math about 50% of the students will say there is no solution and 50% of the students will say there is a solution. So what is the answer? Consider that $0 \cdot 0 = 0$. Hence there is a number that times itself does equal zero. The answer is:

$$\sqrt{0} = 0$$

This means that we have a solution for square roots for any number inside the radical sign that is greater than or equal to zero. This provides a method to determine the domain of a square root function as the set of all numbers which makes inside the radical sign greater than or equal to zero.

Find the domain of $\sqrt{2x + 1}$. To accomplish this, we just need to set the expression under the radical sign greater than or equal to zero and solve:

$$2x + 1 \geq 0$$
$$2x \geq -1$$
$$x \geq -\frac{1}{2}$$

The domain of $\sqrt{2x + 1}$ is $\{x | x \geq -\frac{1}{2}\}$.

Find the domain of the following radicals.

14) $\sqrt{2x - 1}$

15) $\sqrt{-3x + 9}$

16) $\sqrt{x + 12}$

You may have had experience with complex numbers which include both real numbers and imaginary numbers. By imaginary numbers, we mean numbers that include $\sqrt{-1} = i$. Please keep in mind that today we are not using imaginary numbers so that $\sqrt{-1} = i$ is not to be included.

Hence working only in the real number system $\sqrt{-1}$ has no solution.

(If you have not seen imaginary numbers yet, you will see them soon.)
Returning to the example above \( f(x) = \sqrt{2x + 1} \), let’s graph the problem. First let’s create a \( t \)-chart. In choosing values for \( x \), we need to find values that will make inside of the square root sign equal to 0, 1, 4, 9, and 16, since these are perfect squares that we can simplify.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sqrt{2x + 1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1/2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3/2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>15/2</td>
<td>4</td>
</tr>
</tbody>
</table>

Next, we need to plot these points on a graph and draw an approximate curve. Note that the curve does not go to the left of the point \((-1/2, 0)\), as points to the left are outside of our domain.

![Graph of \( f(x) = \sqrt{2x + 1} \)](image)

Obviously, a smooth curve through every point is the actual curve, but our freehand curve is a good approximate curve.

Graph the following square root functions.

17) \( f(x) = \sqrt{2x - 1} \)
A cube root does not have a restricted domain. This is because we can obtain a negative answer when multiplying a negative number together three times so that:

\[ \sqrt[3]{-8} = -2 \]

Create a \( t \)-chart and graph \( f(x) = \sqrt[3]{x + 4} \). In this case, you want to choose \( x \) so that inside the cube root equals -27, -8, -1, 0, 1, 8, and 27, as these are the values where it is easy to determine the cube root.
Graph the following cube root functions.

20) \( f(x) = \sqrt[3]{2x - 3} \)

21) \( f(x) = \sqrt[3]{-3x + 5} \)

22) \( f(x) = \sqrt[3]{x} \)

Final Note about radicals and restricted domains:

- If the index of the radical is even (such as a square root or \( \sqrt{\text{ }, \sqrt{\text{ }, \sqrt{\text{}}} \)) then the domain will be restricted and inside the radical must be greater than or equal to zero – as above for the square root example.
- If the index of the radical is odd (such as a cube root or \( \sqrt[3]{\text{ }, \sqrt[3]{\text{ }, \sqrt[3]{\text{}}} \)) then there is no restriction on the domain – as above for the cube root problem.

Final challenge: Graph the following radical functions.

23) \( f(x) = \sqrt[2]{2x - 3} \)

24) \( f(x) = \sqrt{-3x + 5} \)

25) \( f(x) = \sqrt[3]{x} \)