Quadratic functions are of the form:

\[ f(x) = ax^2 + bx + c \]

where \( a, b, \) and \( c \) are real number coefficients. Last week we worked with moving objects and saw that physics desires the equation to be written in ascending order of degree:

\[ f(x) = c + bx + ax^2 \]

This week we will begin working with projectiles. The primary equation for the height of projectiles is:

\[ h = h_0 + v_0 t - \frac{1}{2} gt^2 \]

This is precisely the same equation we used last week, with \( t \) representing time, and \( v_0 \) representing the initial velocity. The changes from last week begin with using \( h \) to represent height (instead of \( x \) for distance) and \( h_0 \) to represent the initial height. Also, we are using \( g \) to represent acceleration as the acceleration applied on a projectile is gravitational. Also, \( g = 9.8 \, m/s^2 \) as gravitational force is constant. To make the acceleration from gravity pull the body back to the earth, we are subtracting the term representing acceleration.

Assume that we are throwing a ball up in the air from the basement at 3 meters below ground \( (h_0 = -3 \, m) \) at an initial velocity of thirty meters per second \( (v_0 = 30 \, m/s) \). Then the equation for height is:

\[ h = -3 + 30t - \frac{9.8}{2}t^2 \]

Quadratic functions have an interesting graph: Given the function \( h = -3 + 30t - \frac{9.8}{2}t^2 \), complete the \( t \)-chart, graph the points and draw a smooth curve to indicate the complete graph.

\[
\begin{array}{c|c}
  t & h \\
  \hline
  0 & \\
  1 & \\
  2 & \\
  3 & \\
  5 & \\
  7 & \\
\end{array}
\]
This gives the basic shape of the graph of a quadratic function. This graph is unique in that it has a name not associated with the name of the equation, quadratic function. The name of the graph is a parabola. The reason why the two names are not the same is that both were used for several thousand years prior to the realization that they were related. The quadratic function’s primary use prior to the 1600’s was to find the area of four sided shapes and thus earned the name quadratic. As for the shape of the graph, because of its special properties of being able to focus light (glass lenses), the shape became important a long time ago as it is the perfect shape for lenses such as those used in eyeglasses, cameras and flashlights. The name given the shape was a parabola. It wasn’t until we began graphing in the seventeenth century that the relationship between the two became apparent. However, by this time the two names were well grounded in our vocabulary and have not changed.

All quadratic functions graph in the shape of a parabola. This is important to know so that when we are asked to graph a quadratic function we actually graph a function. Consider the following equation, \( f(x) = x^2 + 2x - 1 \). Use the following \( t \)-chart and graph only the points from the \( t \)-chart.

\[
\begin{array}{c|c|c}
\hline
x & f(x) \\
\hline
0 & \ \\
1 & \ \\
2 & \ \\
3 & \ \\
4 & \ \\
5 & \ \\
\hline
\end{array}
\]

If these points are smoothly connected, do we obtain the shape of our first graph completed above? If not, why not? And how do we fix it?

A parabola will have the following features:
1) It may have two x-intercepts found by the quadratic formula: \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), where \( a, b \) and \( c \) are the appropriate coefficients.

2) A vertex: The x-value is at \( -\frac{b}{2a} \), and is at the highest or lowest point of the parabola.

3) A line of symmetry. This is a vertical line through the vertex and if we examine both sides of the graph as separated by the line of symmetry, we will find that the graph is a mirror image about the line of symmetry.

4) A y-intercept at the point \((0, c)\). Let \( x = 0 \) and clearly we will see that the only non-zero value is \( c \).

5) We often say that the graph is a U-shape as this is easy to describe, but what decides if it opens up or down?

Let’s look at these features starting with the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

It determines the x-intercepts. What happens when \( a=1, b=2 \) and \( c=3 \)? In this case \( b^2 - 4ac = -8 \). Placing this value inside the square root sign does not give a real number answer. This implies that there are no x-intercepts. The expression inside the square root is called the discriminate, because it determines the number of zeros:

- If \( b^2 - 4ac > 0 \), then there are two real zeros.
- If \( b^2 - 4ac = 0 \), then there is one real zero and the vertex lies on the x-axis.
- If \( b^2 - 4ac < 0 \), then there are two non-real zeros. And there are no x-intercepts.

\[
f(x) = x^2 + 2x - 3 \quad \quad b^2 - 4ac = 16 > 0
\]

\[
f(x) = x^2 + 2x + 1 \quad \quad b^2 - 4ac = 0
\]

\[
f(x) = x^2 + 2x + 2 \quad \quad b^2 - 4ac = -4 < 0
\]
Find the $x$-intercepts of the following quadratic functions. If there are no $x$-intercepts explain why.

1) $f(x) = 2x^2 + 4x - 1$

2) $f(x) = 3x^2 + 6x - 6$

3) $f(x) = x^2 + 2x - 1$

4) $f(x) = 2x^2 + 4x + 4$

If we look at a function which has two $x$-intercepts, say $f(x) = x^2 + 2x - 3$. We can obtain these two $x$-intercepts by using the quadratic formula, (written in a slightly different fashion):

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Solving for both the plus and minus we obtain the two solutions: $x = -1 + 2$ and $x = -1 - 2$. Work this out for yourself to verify this. This means we are starting at -1 and moving in both directions a distance of 2. Based on the fact that the parabola has a line of symmetry, this would indicate that the line of symmetry is at $x=-1$. Also, based on the fact that the vertex is on the line of symmetry, the $x$-value of the line of symmetry is -1. To find the $y$-value of the vertex, we need to place the -1 in for $x$ into the original equation:

$$f(-1) = (-1)^2 + 2(-1) - 3 = -4$$

Thus the vertex is at $(-1, -4)$.

To recap that last bit of information, to find the vertex, we simply solve for $-\frac{b}{2a}$, and then substitute that back into the equation to find the vertex, $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. 

Find the vertex of the following equations.

5) \( f(x) = 2x^2 + 4x - 1 \)

6) \( f(x) = 3x^2 + 6x - 6 \)

7) \( f(x) = x^2 + 2x - 1 \)

8) \( f(x) = 2x^2 + 4x + 4 \)

Finding the y-intercept is easy, just let \( x = 0 \). This leaves only the value of \( c \).

Find the y-intercept in the form of the point \((0, c)\) for the following functions.

9) \( f(x) = 2x^2 + 4x - 1 \)

10) \( f(x) = 3x^2 + 6x - 6 \)

11) \( f(x) = x^2 + 2x - 1 \)

12) \( f(x) = 2x^2 + 4x + 4 \)
At this time, let’s connect the mathematics of the quadratic equation to solving an ionization problem in chemistry. At 25°C, a 0.0100M NH₃ is 4.1% ionized. Making assumptions of no volume changes calculate the [OH⁻] (concentration of hydroxide) after 0.0090 mol NH₄CL is added to 1 L of the above solution. The chemical equation is:

\[
\text{NH}_3 + \text{H}_2\text{O} \leftrightarrow [\text{NH}_4^+] + \text{OH}^- 
\]

At this point, we hope the reader recognizes this equation as a form of a mathematical representation of the chemical reaction we are currently discussing. A detailed discussion of this problem requires an understanding of the symbols used in chemistry. Thus, we will skip the details and hope the reader is motivated to develop a strong understanding of chemical notation early in their chemistry class.

Through the process of solving this problem and letting \( [\text{OH}^-] \), the following quadratic equation is developed (how this was done will be discussed in chemistry):

\[
x^2 + 0.009175x - (1.75 \times 10^{-7}) = 0
\]

We will leave it to the reader to use the quadratic formula to find the concentration of hydroxide at this time.

How many answers did you obtain? How many make sense? Which answers are in the feasible region and which answers are outside the feasible region and should be ignored?

Additional Practice: We now have the x-intercepts (when they exist), the vertex, the line of symmetry and the y-intercept. Knowing that these equations are Parabolic (U-) shaped, graph the following.

13) \( f(x) = 2x^2 + 4x - 1 \)

14) \( f(x) = 3x^2 + 6x - 6 \)
So far each equation we have been given has “opened upward”. Graph the next set of equations using all of the tools we have discussed so far. Two will “open upward” and two will “open downward”. After completing problems 17-20, state a rule to determine, before graphing, whether the parabola opens up or down.

17) \( f(x) = x^2 + 4x - 1 \)

18) \( f(x) = 3 + 8x - 2x^2 \)

19) \( f(x) = -2 + 4x + 2x^2 \)

20) \( f(x) = -2x^2 + 4x + 4 \)