Today’s topic is logarithms (or in short hand – logs). The concept of logs was initially developed (350 years before calculators) to make multiplication and division easier by transforming multiplication into addition and division into subtraction. Today, the use of logarithms is extensive but due to calculators, logs are no longer used for the primary purpose of turning multiplication into addition and division into subtraction. Hopefully, by knowing the basic premise as to why logs were developed, it will be easier to build an understanding for logs as throughout this activity, the relationship between multiplication and addition (division and subtraction) will be very important.

**Rule of 72**: The rule of 72 is a way of approximating the amount of time it takes for an investment to double given a specific interest rate. Last week we discussed the formula for continuous rates: \( P_t = P e^{rt} \). And today we have been discussing logarithms. Combined, we are now ready to develop the rule of 72.

Since we are interested in doubling our population, let \( P = 1 \) and \( P_t = 2 \).

This gives the equation

\[
2 = e^{rt}
\]

Applying the natural log (ln – see side note) to both sides we have

\[
\ln(2) = \ln(e^{rt})
\]

Since natural logs (ln) and the natural base (e) are inverse function then \( \ln(e^{rt}) = rt \) so we now have

\[
\ln(2) = rt
\]

Divide both sides by the interest rate, \( r \), to find the time to double as

\[
t = \frac{\ln(2)}{r}
\]

Plugging \( \ln(2) \) into the calculator we have \( \ln(2) = .693147108... \approx .693 \). Next remember that when using interest rates, 6% is placed into equations as .06. In this formula, we multiply .693 by 100 and obtain 69.3 and we do not change the percentage to a decimal number thus, when plugging \( r \) into the equation we will plug in 6.

Recall, the inverse to the exponential function, \( e^x \), is \( \log_e x \). however, this button is not on the calculator. The reason is that e is considered to be the natural base and its inverse is the natural log. So \( \log_e x \) is the natural log of \( x \). Due to the numerous opportunities to write \( \log_e x \) in textbooks, a short hand was developed for the natural log. Thus \( \log_e x = \ln x \). And if you look closely on your calculator, you will find a button with ln.
This is one of the few places where percentages are not changes but instead the change has been placed in the equation. So

\[ t \approx \frac{69.3}{r} \]

So why the rule of 72? Why not the rule of 69? Or why the rule of 70? If you do a search for any of these rules, you will find all three exist for the same purpose. Realize that the continuous interest formula used to find the rule of 69 (or 70) is an approximate for compound interest and thus will only produce an approximate answer. The Rule of 72 was first computed based on the compound interest formula and has two advantages over the other two values.

- The approximate answer obtained using the rule of 72 is closer to the correct value in the range of 6% and 10%.
- The number of factors of 72 (1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72) makes dividing into 72 quickly much easier than quickly dividing into 69 or 70.

There you have a quick and easy method for computing the time for an investment to double based on exponential and logarithmic functions.

In using the rule of 72 in science, consider that a few weeks ago, you worked with both rates of growth and decay problems. Here you will also work with a decay problem called half-life. If you are given the rate, you can find the half-life (doubling time) or if you are given the half-life (doubling time) you can find the approximate rate.

If the growth rate is 8% then the doubling time is \[ \frac{72}{8} = 9 \text{ years} \].

If the half-life for carbon-14 is 5370 years, then the rate of decay is \[ -\frac{72}{5370} = -0.013\% \]. (Negative sign because there is a decrease in size.)

Given the following growth rates, find the approximate doubling time:

1. 3%
2. 5%
3. 7.1%
Given the following half-lives, find the approximate decay rate. (In this case realize that the time given is important. If the rate is 7% and the time is in days, this means that 7% is lost each day. If the time is in years, this means that 7% is lost each year.):

1. The half-life of bismuth-210 is 22 days.

2. The half-life of polonium-210 is 5 days.

3. The half-life of lead-214 is 3.1 minutes.

In examining the rule of 72, we used the inverse property, \( \log_b b^x = x \) to bring the variable out of the exponent. We use the other inverse property, \( \log_b b^x = x \), to bring the variable out of a log as follows: Find \( x \), given

\[
\log x = 2.3
\]

\[
10^{\log x} = 10^{2.3}
\]

\[
x = 10^{2.3}
\]

\[
x = 199.52...
\]

Use the inverse properties to solve the following:

1. \( \log x = 1.4 \)

2. \( \log x = 15 \)

3. \( 10^x = 6 \)

4. \( e^x = 6 \)

5. \( \ln x = 2 \)
Let’s review four more properties of exponents.

\[ \log_b 1 = 0 \]

\[ \log_b x + \log_b y = \log_b (xy) \]

\[ \log_b x - \log_b y = \log_b \left(\frac{x}{y}\right) \]

\[ \log_b x^a = a \log_b x \]

Combine the following problems into one log:

1. \[ \log_2 m + \log_2 n + \log_2 p \]
2. \[ \log_4 x + 2 \log_4 y + \log_4 1 \]
3. \[ \ln x - \ln y + 2 \ln z \]

Separate so that each letter is written in a separate log. This implies that you are doing the reverse action as completed in the last three problems.

1. \[ \log(xyz) \]
2. \[ \log \frac{x^2 y^3}{\sqrt{z}} \]
3. \[ \log \frac{x^3}{y z^4} \]

Final thought on logs. When examining your calculator you only have two buttons with respect to logs – \( \log_{10} \) and \( \ln \). What happens if you need to find \( \log_3 14 \)? To solve this, we will develop a change of base formula using the log properties listed above.

\[ \log_b x = y \]

\[ b^{\log_b x} = b^y \quad \text{Raise both sides to the base, } b \]

\[ x = b^y \quad \text{Use inverse property to simplify} \]

\[ \log_{10} x = \log_{10} b^y \quad \text{Apply log to both sides} \]

\[ \log_{10} x = y \log_{10} b \quad \text{Use exponent property to bring the } y \text{ out front} \]

\[ \frac{\log_{10} x}{\log_{10} b} = y \quad \text{Divide both sides by } \log_{10} b \]
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This gives us a very simple change of base formula: \( \log_b y = \frac{\log y}{\log b} = \frac{\ln y}{\ln b} \)

Since your calculator has both the ability to do log (\( \log_{10} \) or log base 10) and ln, you may use either one when changing the problem.

Thus \( \log_3 14 = \frac{\log 14}{\log 3} \approx 2.4021 \ldots \)

Find the following logs:
1. \( \log_6 22 \)
2. \( \log_3 2100 \)
3. \( \log_4 1570 \)
4. \( \log_5 13 \)

Closing out with Rate law and First-order Chemistry Equations:
Zero-order chemistry reactions and linear equations for this type of reaction take on the form:

\[
[A]_t = -kt + [A]_0
\]

This week we desire to introduce first-order reactions. First-order reactions, are reactions where amount of the chemical compound in the reaction decreases directly proportional to the concentration. And the equation is

\[
\ln[A]_t = -kt + \ln[A]_0
\]

Note: \( \ln \) in the notation \( \ln[A]_t \) represents the natural log.

Given the time and concentration listed in the chart, compute the third column which is the natural log of the concentration. Then create a graph of time versus \( [A] \) and a second graph of time versus \( \ln[A] \)

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>([A])</th>
<th>(\ln[A]) Fill in this column</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.074</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.044</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.027</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.0013</td>
<td></td>
</tr>
</tbody>
</table>
What type of function would say that each of the graphs above represents (linear, quadratic, rational, exponential, etc.)? Explain why you choose the type you did.

It should be noted that by changing the concentration, \([A]\), value into the new value using the natural log, \(\ln[A]\), the graph becomes linear. Furthermore, the slope of the line is equal to the constant in the rate equation (-k). This is the same the idea that you saw in the logistic activity in week 11 where the slope of the line was the same as the interest rate.