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Video Case Analysis of Students' Mathematical Thinking: Initial Development Process

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Abstract

Over the last 20 years, researchers have explored the use of video for teacher education and as a reflective tool for studying instruction. We are interested in how video might be used to support mathematics teacher candidates' understanding of student thinking. In particular, we are developing and investigating the use of video-based online learning modules as a component of prospective secondary mathematics teacher preparation with the overarching goal of improving their ability to recognize and make connections between patterns in students' informal and formal reasoning. In this paper, we describe the initial development process of video case analysis of student thinking (VCAST) modules for use in an online setting and then the share results associated with our first two modules on functional reasoning. Our design-based development process is informed by literature on student thinking and involves a purposeful selection of authentic examples of student reasoning. The results indicate that our process led to an adequate range in student ideas from which to build the modules. Moving forward, the research team will conduct a series of design experiments to investigate the continued development, improvement, and implementation of these online modules. In a period where programs are looking for ways to integrate innovative uses of technology in the subject matter preparation of teachers, it is critical to have empirically-based models for such learning.

Instruction which progressively builds upon students' ideas increases the accessibility of mathematical ideas to a broader range of students and results in a more equitable practice (Boaler, 2004). Further, when teachers build upon students' thinking to develop their understanding of formal mathematical ideas (e.g. Carpenter, Franke, Jacobs, Fennema, & Empson, 1998), students' conceptual understanding of mathematics is improved while procedural fluency is maintained (e.g., Fennema et al., 1996). As teacher educators, we are aware of the limitations associated with developing teacher candidates' knowledge of student thinking through field-based work alone (Ko, Milewski., & Herbst, 2017). These limitations motivated us to develop and investigate the use of video-based online learning modules to prepare prospective secondary mathematics teachers (PSMTs) to recognize and make connections between patterns in students' informal and formal reasoning. Our intent is to better position prospective teachers to enact classroom practices that build upon individual students' ideas. The purpose of this paper is to describe the initial development process of using video case analysis of student thinking (VCAST) to develop online modules and to share results associated with the development of two modules on functional reasoning.

Background

As teacher educators, we are aware of the importance of PSMTs noticing and making connections between the different ways students reason about mathematics. For a number of years, we had PSMTs either read about students' mathematical ideas or conduct student interviews. We found, though, that both strategies rarely resulted in the PSMTs noticing and making connections between the different mathematical ideas students expressed. We attribute this, in part, to student reasoning examples in readings often focusing on elementary level math and not on the specialized knowledge we sought to develop in the PSMTs. This led us to wonder how video might provide a meaningful context for engaging PSMTs in examining the mathematical ideas expressed by secondary students. We set out two years ago to develop a series of online modules based on video case analysis of student thinking (VCAST). We have since been iteratively testing and updating these modules in authentic settings with PMSTs.

Theoretical Perspective

In this section, we briefly describe the four areas of work closely aligned to our project: (1) specialized knowledge of mathematics for teaching, (2) functional reasoning, (3) video and teacher education, and (4) design-based research.

Specialized Knowledge of Mathematics for Teaching. To advance student understanding, teachers must pay careful attention to students' mathematical ideas and respond to students in ways that enable them to build upon their ideas (Ball, Thames, & Phelps, 2008). Learning to attend and respond in this way requires a specialized knowledge of how mathematical ideas are related, how to represent ideas in meaningful ways, and common patterns in students' reasoning (Ball, Thames, & Phelps, 2008). Without this type of knowledge, teachers' attention to students' ideas can result in teachers doing the work for students (if the student erred) or congratulating the student (if the student used a correct strategy), rather than pressing the student to reformulate or extend their ideas (Henningsen, & Stein, 1997).

Functional Reasoning. A function in mathematics is defined as "a correspondence between two nonempty sets that assigns to every element of the first set (the domain) exactly one element in the second set (the codomain)" (Vinner & Dreyfus, 1989, p. 357). Function is a unifying concept in K-12 mathematics that applies to the study of algebra, geometry, probability, and statistics. However, many secondary students leave high school with impoverished reasoning abilities with functions, equating functions with a single rule or equation and experiencing difficulty in generalizing functional relationships between quantities (e.g. Carlson, 1998; Thompson, 1994).

Understanding a concept in mathematics is achieved when one is able to use, identify, apply, generalize and create extensions of that concept (Sierpinska, 1992). A robust form of functional reasoning is evidenced by the ability to work with functions in a variety of ways, including:

1) using function notation,
2) recognizing a function as a dynamic process that maps each element from a set of inputs to a single element in the set of outputs,
3) building and working with functions as mathematical models of relationships between quantities,
4) interpreting and making connections between multiple representations of functions,
5) coordinating changes in quantities that covary,
6) manipulating functions as abstract objects, and
7) identifying and using structural generalizations for families of functions.

The modules we are developing are designed to address items 1) through 5).

Video and Teacher Education. Over the years, teacher education programs have placed a greater emphasis on field experiences for prospective teachers (Zeichner, 1981). One of the benefits is the opportunity for prospective teachers to see what teaching and learning look like in real classrooms (Wilson, Floden, & Ferrini-Mundy, 2001). As beneficial as field experiences can be, the variance in their quality makes it difficult to concentrate on specific "teachable moments." As a result, teacher educators have explored different ways to use video in teacher education to help slow down and capture these teachable moments (e.g. Sherin & van Es, 2005). In addition, videos of teachable moments can help prospective teachers not only "see" important events but also pause, reflect, and even rewatch them (Coffey, 2014). Our use of video to prepare teachers has been influenced by the work of multiple researchers (e.g., Borko, Jacobs, Eiteljorg, & Pittman, 2008; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012).

Design-based Research. Technology-based interventions often fail. This usually happens when educators or developers expect technology alone to fix educational problems. Research has consistently shown that technology is not a panacea; it is the pedagogy and specifically instructional design that makes a difference in learning outcomes (Blomberg, Sherin, Renkl, Glogger-Frey, & Seidel, 2014). Following Lesh, Kelly, & Yoon (2008), we have set out to design, develop, and improve the use of our modules—and in turn PSMT learning—through an iterative cycle of design experiments that involve (1) conjecture-building, (2) module (re)design, (3) module piloting, and (4) analysis and reflection on related conjectures. Design research methodology (Anderson & Shattuck, 2012; Collins, Joseph, & Bielaczye, 2004; Reeves, Herrington, & Oliver, 2005) is particularly applicable for curriculum development as it provides "an avenue for studying learning within the complexity of interacting educational systems" (Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008, p. 220) and has been effectively used in mathematics education to investigate student learning (c.f. Lobato, 2008) and teachers' development of practice (c.f. Zawojewski et al., 2008).

Video Case Analysis of Student Thinking (VCAST) Modules

Each online module incorporates short video clips of secondary students working on a mathematical task and a series of questions focused on the students' mathematical reasoning while solving the task. The overarching learning goal with each module is for PSMTs to recognize and make connections between secondary students' informal and formal reasoning about key ideas of functions and modeling. The completed modules feature students working with figural patterns (Module I) and graphs of discrete data (Module II).

The modules are designed to be used in various formats. While we are specifically interested in using them in face-to-face or hybrid undergraduate courses with the intent that PSMTs complete each module before engaging in a face-to-face follow-up session, they could also be used in fully online courses. Currently, the modules are embedded in an upper-division mathematics course. The general structure of each module engages PSMTs in a cycle of different design elements. First, PSMTs solve the mathematical task and provide a written explanation of their work. Next, PSMTs either examine written student work or watch short video clips of students working on the task. Then, based on the evidence provided, PSMTs are asked to formulate hypotheses about the secondary students' reasoning about key ideas of functions and modeling. Once these hypotheses are made, additional student evidence is provided, either written work or subsequent video clips. PSMTs then have the opportunity to revise their initial hypotheses about the students' reasoning. Finally, the PSMTs are prompted to reflect on what they have learned about the mathematics of the task and to report on any new insights they have gained.

After completion of the module, and prior to the follow-up session, PSMTs read a research-based article that addresses patterns in students reasoning on tasks related to the one featured in the module. During follow-up sessions, PSMTs work in small groups to compare observations about student thinking from the module and to identify explicit connections between student approaches. At the conclusion of the follow-up session, PSMTs engage in a whole group discussion about connections to the assigned reading and address approaches that may not have been evident in the module's featured video cases.

Module I: The Hexagon Task. Figural pattern tasks such as the hexagon task in Figure 1 can be used to elicit students' ideas about building functions that model a situation. Figural patterns allow students to make algebraic generalizations in a variety of ways, potentially leading to the meaningful use of variables and expressions with the quantities involved. In particular, near generalization tasks can be solved by step-by-step drawing or counting (e.g., "What is the perimeter of the 5th figure in the hexagon task below?"), whereas far generalization tasks require determining a pattern that can be quickly applied to a larger number, such as the hexagon task in Figure 1 (Stacey, 1989).

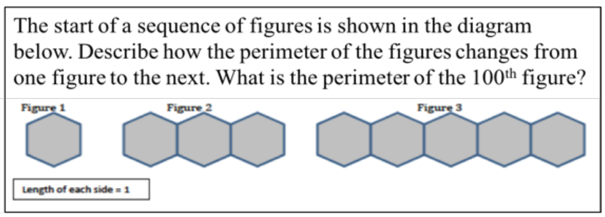

Figure 1. The Hexagon Task (Adapted from Hendrickson, Honey, Juehl, Lemon, & Sutorios, 2012).

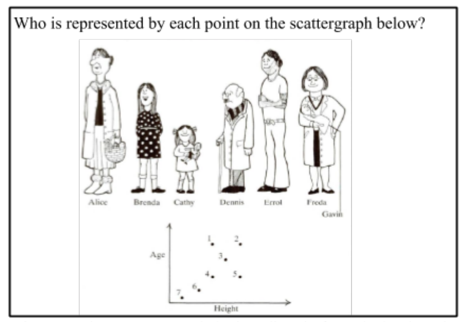

Figure 2. The Bus Stop Task (Taken directly from the book, "The Language of Functions and Graphs," published by the Shell Centre for Mathematical Education.)

Module II: The Bus Stop Task. Graphing tasks such as the bus stop task in Figure 2 can be used to elicit evidence of students' covariational reasoning. As noted in various research studies focused on the development of functional reasoning (Confrey & Smith, 1995; Oehrtman, Carlson & Thompson, 2008; Saldanha & Thompson, 1998), students' ability to coordinate the relationship between successive values in two sequences and then to couple two quantities into a singular object that can be represented graphically is both developmental and nontrivial. In this task, students are asked to coordinate the static quantities of height and age for a group of seven individuals presented pictorially and then to represent this coordination in a Cartesian graph.

Methods for Initial Development

The initial development process for a VCAST module involves three stages (See Table 1). In the first stage, we identified a key mathematical learning goal associated with student thinking, consulted the related literature on student reasoning, and selected a task that could be used with secondary students to elicit a range of thinking. An important consideration in this first stage was research evidence on how student thinking progresses. In the second stage, researchers conducted one-on-one interviews with secondary students, analyzed the student work (evidenced by video & student artifacts) in relation to the literature on student thinking and developed codes relevant to a hypothesized progression of learning. Qualitative analysis of student work involved establishing a set of codes for categories of reasoning that were independently applied by at least two researchers. Discrepancies in coding were negotiated among all the researchers until agreement was reached. Depending on the task, the list of codes for a task was either clearly informed by the literature on student thinking (e.g. Rivera, 2010) or developed through a grounded theory approach (Strauss & Corbin, 1994) of creating and revising codes based on categories of reasoning anticipated by the researchers. In the third stage, we selected a range of representative samples to potentially use for video cases, independently developed a proposed module structure, and then worked together to blend selected elements from the proposed module structures. Module learning goals were drafted at the beginning of the process and then revised.

Table 1: Initial Development Process Stages

Stage 1: Task Selection	**Stage 2: Interview & Analyze**	**Stage 3: Design Prototype**
Identify mathematical learning goal associated with student thinking	Capture student thinking during one-on-one interviews	Select potential video cases associated with the learning progression
Review related literature on student reasoning	Analyze interview data in relation to the literature on student reasoning	Independently develop three possible module structures for consideration.
Select a task that is aligned to the goal and has potential to elicit a range of student thinking	Develop codes relevant to a progression of student learning	Blend desired elements from drafted module structures and revise learning goals for PSMTs.

Results and Discussion

The interviews with secondary students were designed to capture a range of student thinking. We worked directly with classroom teachers to select students for interviews who would be able to articulate their thinking in an interview setting. We conducted and recorded one-on-one interviews with 12 high school calculus students and 23 middle school students in 7th and 8th grades. The hexagon task was one of several tasks completed by the high school students. The bus stop task was one of three tasks completed by the middle school students. As students worked through each task, a researcher asked students to document their work on paper and to explain their thinking aloud. The researcher also asked questions to support student progress. Each interview lasted about 20 minutes.

The Hexagon Task Results. Approaches to figural pattern tasks can be described in relation to four interrelated categories that depend on the representations students choose for the elements in the sequence (Sequence Elements) and their symbolic generalizations (Symbolic Generalization). (See Figure 3). With respect to the hexagon task, students may approach the problem by generating a numerical sequence that corresponds to the perimeter of the figure for the first several figures and then base their reasoning on patterns they notice in the numerical sequence. When doing so, they are using a numerical representation of the sequence elements. They may then generalize patterns either by stating a recursive pattern that describes how the values change from one element to the next (Numerical & Recursive), or a functional pattern that describes how to generate a sequence element based on the sequence position (Numerical & Functional). When students observe patterns related to the geometric features of the figures, they are using a figural representation of the sequence elements. These patterns may focus on how the hexagon figures change from one figure to the next (Figural & Recursive), or how the perimeter can be determined based on either the number of hexagons or the figure number (Figural & Functional).

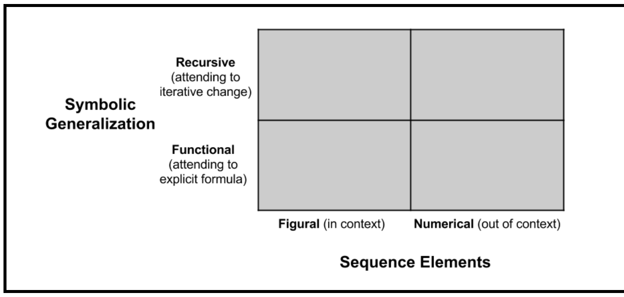

Figure 3. Figural pattern approaches

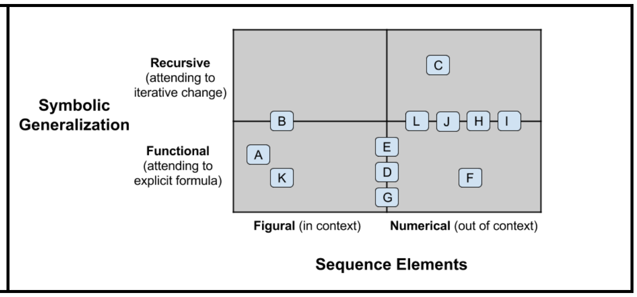

Figure 4. High school student approaches (*n*=12)

The high school students we interviewed used a variety of approaches when solving the hexagon task. Figure 4 illustrates the range of approaches students demonstrated. When solving the hexagon task, some students reasoned through the task using one of the four approaches (Students A, C, F, K) or some combination (Students B, D, E, G, H, I, J, L). For example, one student noticed the number of hexagons increases by two for each figure and then used that information to determine a functional relationship for the perimeter in terms of the number of hexagons (Student B). Other students started by observing the increase in perimeter for each figure and then used that information to generate an explicit function for the perimeter of each figure (Students H, I, J, L). And other students started by trying to write a function for the perimeter based on the number of hexagons (observing how the hexagons in each figure contribute to the perimeter) and then used a numerical approach to determine the number of hexagons in each figure (Students D, E, G). What is important is that there are identifiable patterns in how the students reasoned about the task that can be built upon in instruction.

As we conducted our analyses, we developed a list of potential opportunities for PSMT learning: (1) deciphering students' cryptic notations; (2) recognizing what is correct in a slightly flawed strategy, (3) recognizing ideas related to an efficient strategy, and (4) making connections between different strategies. We also wanted the module to introduce PSMTs to a range of ways to approach the hexagon task, but not necessarily all possibilities. We suspected that PSMTs would be familiar with a numerical approach and less familiar with figural approaches.

The list of conjecture opportunities for learning, together with the analyses of students' approaches, led to the articulation of learning goals for the module. Specifically, Module I is designed to support PSMTs' ability to:

- Listen and attend to students' mathematics as they make generalizations about a sequence of geometric figures
- Articulate students' mathematics about figural and numerical generalizations for a given sequence and how those generalizations are reflected in students' work
- Recognize correct reasoning expressed via informal language and vocabulary that describes figural and numerical generalizations
- Describe mathematical connections between PSMTs' mathematical approaches and students' approaches

These learning goals were articulated and revised in the process of developing the module content.

We selected the work of Student A (Figural & Functional) and Student J (Numerical & Recursive and Numerical & Functional) from which to build case studies for Module I. Because we anticipated PSMTs would be less familiar with Student A's approach, we put Student J's case study before Student A's in the module. Recall, PSMTs work through the same task at the beginning of the module. Thus, starting with student work that is potentially more familiar should facilitate PSMTs' progress through the first part of the module. We structured each case study in a similar way, starting with the PSMTs making a hypothesis about the student's approach to the hexagon task based on either a short video clip of the student sharing their thinking (Student J) or based upon an image of the student's work (Student A). Each case study was designed to then present additional information about the students' reasoning and prompt PSMTs to revise their hypotheses. The case studies conclude with a prompt for PSMTs to compare their own approach to the hexagon task to the student's approach.

The Bus Stop Task Results. Student approaches to the bus stop task can be categorized by the evidence they produce related to four key conceptual understandings: (1) ordering quantities without assigned values, (2) utilizing graphical conventions as they relate to the assigned meaning of horizontal locations in the Cartesian plane, (3) utilizing graphical conventions as they relate to the assigned meaning of vertical locations in the plane, and (4) connecting the coordination of paired covarying quantity values to their representations in the plane.

As anticipated, our interviews of middle school students revealed a range of thinking about the bus stop task. Table 2 provides a summary of our analysis of the collected video and written student data. One student was unable to detach the meaning of a point's location on the plane from its numerical label and vacillated between having the label represent height or age. Other students (n = 4) attended solely to height and ordered the people represented in the task accordingly. When they encountered two individuals with the same height, they either used age or their position in the picture to "break the tie." In contrast, those who exhibited nascent covariational reasoning (n = 19) appeared to understand that a single point in the plane carried information about paired quantities. They illustrated this understanding by attending to both height and age as they matched individuals to graphed points. Those who were able to maintain a stable assignment of quantity to graphical coordinate (n = 7) were successful with the task, though 2 of these 7 corrected their thinking following an interviewer's clarifying question about their approach. Others, for whom the quantity to coordinate assignment was less stable (n = 12), were successful with the task until they encountered individuals or points with a shared quantity value. A trend that emerged from our analysis indicates that assigning quantities typically represented vertically (such as height) to a horizontal coordinate in a graphical representation is challenging for middle school students.

Table 2: Bus Stop Task Summary of Student Reasoning (n = 23)

Conceptual Understanding	Brief Description as it Relates to the Bus Stop Task	Coding Options and Percentage Receiving Code
A single point can represent the values of two different quantities.	Student understands that each point in the Cartesian plane supplies both age and height information.	Not demonstrated (n = 4)
		Partially demonstrated (n = 4)
		Clearly demonstrated (n = 15)
One-dimensional graphical	Student understands that moving up	Never (n = 5)

conventions apply to all vertical axes or number lines.	corresponds to increasing age, while moving down corresponds to decreasing age.	Sometimes correct (n = 11)
		Always correct (n = 7)
One-dimensional graphical conventions apply to all horizontal axes or number lines.	Student understands that moving right corresponds to increasing height, while moving left corresponds to decreasing height.	Never (n = 4)
		Sometimes correct (n = 12)
		Always correct (n = 7)
Quantities do not require numerical values to be relatively positioned along an unscaled axis.	Student can compare and order relative heights and/or ages that do not have assigned numerical values.	Not demonstrated (n = 1)
		Clearly demonstrated (n = 22)

While we conducted our analyses, we again developed a list of potential opportunities for PSMT learning as it relates to recognizing (1) differences between students' bivariate and univariate reasoning, (2) the evidence of reasoning students' gestures and verbal explanations provide, (3) how features of tasks can elicit student evidence with potential to be misinterpreted, and (4) how assignment of height quantities to horizontal coordinates can reveal weaknesses in students' covariational reasoning flexibility. We also wanted the module to reveal the challenges inherent to gauging the difficulty of a task.

This list, combined with our analysis of student approaches, led to the articulation of learning goals for the module. Specifically, in the process of developing the module content, Module II was designed to support PSMTs' ability to:

- Listen and attend to students' mathematics as they reason about bivariate data associated with a static situation involving unmeasured quantities.
- Articulate students' mathematical reasoning about the relative location of points in two dimensions within a context.
- Recognize correct reasoning expressed via gestures and informal language that describes how the relative location of points in two dimensions can be used to draw contextual conclusions.
- Recognize how features of a task can influence its difficulty and impact the reasoning students exhibit.

We selected the work of three different middle school students for Module II's case studies. Because we anticipated that PSMTs would struggle to accurately gauge the task's difficulty, the module asks them to identify components of the task they deem easy and challenging and to rank the difficulty on a scale from 1 to 10. We also anticipated that students' initial assignment of points 7, 6, and 5 might lead PSMTs to make faulty assumptions about students' understanding. For this reason, we provided three video clips which feature students starting the task in different ways and then ask the PSMTs to predict who will be the most successful. We then provide the subsequent video clips for these same three students so PSMTs can revise their initial hypotheses and also to highlight the ways in which a student can (a) reason correctly and still arrive at an incorrect answer when one of the unmeasured quantities (age) is challenging to estimate, (b) correct errors while progressing through a task, and (c) reason correctly but then falter when cognitive disequilibrium is triggered (vertical measures such as heights can be assigned to horizontal coordinates in the plane). At the end of the module, PSMTs are again asked to rank the task's difficulty and to make a prediction about each student's approach to a revised version of the task.

Conclusion

Our project illustrates one way to purposely design video-based instructional modules informed by literature on student thinking and built around authentic examples of student reasoning. To iteratively improve the VCAST modules, the research team will conduct a series of design experiments using mixed methods to investigate the development, improvement, and implementation of online modules focused on engaging PSMTs in recognizing and making connections between patterns in students' mathematical reasoning. In a period where programs are looking for ways to integrate subject matter preparation with learning about teaching practices, it is critical to have empirically-based models for such learning.

References

Anderson, T., & Shattuck, J. (2012). Design-based research a decade of progress in education research? *Educational researcher*, *41*(1), 16-25.

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Blomberg, G., Sherin, M. G., Renkl, A., Glogger-Frey, I., & Seidel, T. (2014). Understanding video as a tool for teacher education: Investigating instructional strategies to promote reflection. *Instructional Science, 42*(3), 443-463.

Boaler, J. (2004). *Promoting equity in mathematics classrooms–important teaching practices and their impact on student learning*. Paper presented at the International Congress on Mathematical Education.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and teacher education*, *24*(2), 417-436.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education III, 7, 114–162.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Coffey, A. M. (2014). Using video to develop skills in reflection in teacher education students. *Australian Journal of Teacher Education*, *39*(9), 86-97.

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *The Journal of the Learning Sciences*, *13*(1), 15-42.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26,*66-68.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Joint Matriculation Board and Shell Centre for Mathematical Education. (1988). *The language of functions and graphs: an examination module for secondary schools*. Harlow: Longman.

Kersting, N. B., Givvin, K. B., Thompson, B. J., Santagata, R., & Stigler, J. W. (2012). Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning. *American Educational Research Journal*, *49*(3), 568-589.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 524-549.

Ko, I., Milewski, A., & Herbst, P. (2017, April). *How are pre-service teachers' educational experiences related to their mathematical knowledge for teaching geometry?* Paper presented at the annual meeting of the American Educational Research Association Annual Meeting. San Antonio, Texas.

Lesh, R. A., Kelly, A. E., & Yoon, C. (2008). Multitiered design experiments in mathematics, science, and technology education. In A. E. Kelly, R. A Lesh, & J. Y Baek (Eds.), *Handbook of design research methods in education* (pp. 131-148). New York, NY: Routledge.

Lobato, J. (2008). Research methods for alternative approaches to transfer: Implications for design experiments. In A. E. Kelly, R. A Lesh, & J. Y Baek (Eds.), *Handbook of design research methods in education* (pp. 167-194). New York, NY: Routledge.

Oehrtman, M., Carlson, M. & Thompson, P.W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education*. Washington, DC: Mathematical Association of America.

Reeves, T. C., Herrington, J., & Oliver, R. (2005). Design research: A socially responsible approach to instructional technology research in higher education. *Journal of Computing in Higher Education*, *16*(2), 96-115.

Rivera, F. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics, 73,* 297-328.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson, K. R. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Sherin, M. G., & van Es E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education*, *13*(3), 475-491

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function aspects of epistemology and pedagogy* (pp. 25-58). Washington, DC: Mathematical Association of America.

Stacey, K. (1989). Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics, 20*, 147-164.

Straus, A., & Corbin, J. (1994). Grounded theory methodology. In N. Denzin & Y. Lincoln (Eds.), *Handbook of qualitative research* (pp. 273-285). Thousand Oaks, CA: Sage.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld & J. J. Kaput (Eds.), *Research in collegiate mathematics education, 1* (Issues in Mathematics Education Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20,* 356-366.

Wilson, S., Floden, R., & Ferrini-Mundy, J. (2001). *Teacher preparation research: Current knowledge, gaps, and recommendations.* University of Washington: Centre for the Study of

Teaching and Policy.

Zawojewski, J., Chamberlin, M., Hjalmarson, M. A., & Lewis, C. (2008). Developing design studies in mathematics education professional development: Studying teachers' interpretive systems. In A. E. Kelly, R. A Lesh, & J. Y Baek (Eds.), *Handbook of design research methods in education* (pp. 219-245). New York, NY: Routledge.

Zeichner, K. M. (1981). Reflective teaching and field-based experience in teacher education. *Interchange*, *12*(4), 1-22.