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Title: Über Folgen stetiger Funktionen
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In this paper Hurewicz considers sequences of continuous real-valued functions on metrizable spaces M. For such a sequence $(f_n : n < \infty)$ of continuous real-valued functions he considers the subset $\{(f_n(p) : n < \infty) : p \in M\}$ of ${}^{\omega}\mathbb{R}$ of sequences of real numbers. This is called the value set of the sequence of functions. He defines the usual notions of being bounded and of not being a dominating family in the eventual domination order on ${}^{\omega}\mathbb{R}$.

In section 2 (p. 194) Hurewicz observes:

A) If M is a union of countably many compact subsets, then the value-set of any sequence of continuous real-valued functions on M is bounded.
In section 3 (p. 194) he gives an example that

B) If M is the set of irrational real numbers between 0 and 1 then there is a sequence of continuous functions on it whose value set is a dominating family.
And for each n he defines the function f_n's value at x, $f_n(x)$, to be the n-th term in the continued fraction expansion of x. He observes that each f_n is continuous and that the value set of this sequence is the set of all ω- sequences of natural numbers. Then he states that from these observations the question arises: What internal properties of a set M are characterized by the fact that the value set of any sequence of real-valued continuous functions on it is bounded (respectively not dominating)?
In section 4 (p. 195-6) he introduces covering properties E* and E**.
In the current SPM notation, E* $= S_{fin}(O, O)$ and E** $= U_{fin}(O, \Gamma)$, where O denotes the collection of open covers of a space, and Γ denotes the collection of open γ covers. Thus E* is also what is called the Menger property, and E** is what is called the Hurewicz property. Then Hurewicz proves

C) $S_{fin}(O, O) = S_{fin}(\Lambda, \Lambda)$
and he observes that

D) $S_{fin}(O, O)$ is preserved by continuous images (Footnote 2, page 195).
In footnote 1 on p. 196 Hurewicz states that evidently E** implies E*, and notes that it is an open problem whether there is a set with property E* but not property E**. Then an additional remark is added during the corrections stage of the paper ("Zusatz bei der Korrektur"): In the remark Hurewicz proves:

E) If M is a subset of a separable metric space and has property E** but empty interior in the space, then M is of first category.
This might be due to Sierpinski, on account of the following. Hurewicz states that Sierpinski observed that if the Continuum Hypothesis (CH) holds then the open problem has a positive answer. The argument is as follows: The Lusin set has property E*, but because of E), does not have property E**.

F) A Lusin set has property $S_{fin}(O, O)$ but not property $U_{fin}(O, \Gamma)$.
In section 5 (p. 196 - 199) Hurewicz proves the following theorem:

G) For a separable metric space M the following are equivalent:

(1) M has property $S_{fin}(O,O)$ (respectively $U_{fin}(O,\Gamma)$).
(2) Every sequence of continuous real-valued functions on M has a non-dominating (respectively bounded) value set.

and then adds

H) For a separable metric space M the following are equivalent:

(1) M has property $S_{fin}(O,O)$.
(2) For any sequence $(f_n : n < \infty)$ of continuous real-valued functions on M, if the sequence a of real numbers does not belong to the value set of $(f_n : n < \infty)$, then there are sequences β and γ of real numbers such that
 (a) $\beta < a < \gamma$ and
 (b) No ξ in the value set of $(f_n : n < \infty)$ satisfies $\beta < \xi < \gamma$.
 (For this property Hurewicz uses the terminology "the value set of $(f_n : n < \infty)$ is closed".)

In Section 6 (p. 199 - 200) Hurewicz discusses Menger's Conjecture that in metrizable spaces $S_{fin}(O,O)$ implies σ-compactness. He reminds the reader that in his 1925 paper he proved that this conjecture is true for analytic sets, and then remarks that the characterization in G) above characterizes the σ-compact sets among the analytic sets as those on which the value sets of a sequence of continuous real-valued functions is bounded. He also observes that Sierpinski showed that the Continuum Hypothesis implies the negation of Menger's Conjecture. Then, on page 200 Hurewicz conjectures that $U_{fin}(O,\Gamma)$ is equivalent to σ-compactness.
In Section 7 (p. 201 - 202) Hurewicz connects this study with problems of Hausdorff -whether there is an unbounded sequence of ordertype ω_1 in the eventual domination order on sequences of reals, or such a dominating sequence. Though Hurewicz considers the first uncountable cardinal number, his next result adapts directly to the modern dominating number and bounding number:

I) The minimal cardinality of an unbounded set in ${}^{\omega}\mathbb{R}$ is the same as the minimal cardinality of a separable metric space without property $U_{fin}(O,\Gamma)$.
and he states that quite analogously one proves

J) The minimal cardinality of a dominating set in ${}^{\omega}\mathbb{R}$ is the same as the minimal cardinality of a separable metric space without property $S_{fin}(O,O)$.
Supplementary Section (p. 202 - 204) In this supplementary section Hurewicz proves:

K) For a metrizable space M the following are equivalent:

(1) M has property $U_{fin}(O,\Gamma)$.
(2) Each metrizable continuous image of M is a union of countably many totally bounded sets.

Thus, a subspace of a metrizable space has property $U_{fin}(O,\Gamma)$ if, and only if, it is sigma-totally bounded. He then reformulates the question behind his conjecture. If a metrizable space has property (2) of K), must it be σ-compact? And in the footnote on page 204 Hurewicz notes that similarly:

L) For a metrizable space M the following are equivalent:

(1) M has property $U_{fin}(O,O)$.

(2) Each metrizable continuous image of M is a union of countably many sets whose diameters converge to zero.

Remarks:

1) Notation: Menger used the symbol E to denote a basis property which he introduced in his 1924 paper. In his 1925 paper where Hurewicz studied Menger's basis property, he used the symbol E* to denote a covering equivalent to Menger's basis property. In the current SPM notation, the property E* is denoted $S_{fin}(O, O)$ where O denotes the collection of open covers of a space. And Hurewicz introduced in the 1925 paper a second covering property which was denoted by the symbol E**. In the current SPM notation, property E** is denoted $S_{fin}(\Omega, O^{gp})$.

2) Hurewicz was familiar with the 1909 papers by Hausdorff on the eventual domination order, and with Hausdorff's 1917 monograph "Set Theory".