$\mathcal{SPM}$ BULLETIN

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1. Editor's note

In addition to the interesting research announcements, we would like to draw your attention to the conference announced in [§3.1 below], in which special sessions will be devoted to infinite games in topology and in set theory, both of interest to readers of this bulletin. Marion Scheepers is an invited speaker in this conference, and I also hope to be able to attend. It would be nice to meet some of you there (that is, in a non-electronic manner).

Email addresses. Because of the rapidly growing problem of spam emails, where the addresses are sometimes found by programs surfing in the internet, we decided to stop, from now on, giving the email addresses of contributors. If you wish to contact a specific author, please email me and I will send his email to you personally.

Contributions to the next issue are, as always, welcome.

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2. Research announcements

2.1. **Proceedings of SPM Workshop.** The proceedings of the *Lecce Workshop on Coverings, Selections and Games in Topology* (June 2002) are going to be published in *Note di Matematica*, volume **22**, no. 2 (2004).

Cosimo Guido

2.2. **A brief remark on van der Waerden spaces.** We demonstrate that Martin's axiom for σ-centered notions of forcing implies the existence of a van der Waerden space that is not a Hindman space. Our proof is an adaptation of the one given by M. Kojman and S. Shelah that such a space exists if one assumes the continuum hypothesis to be true.

`http://www.ams.org/journal-getitem?pii=S0002-9939-04-07351-4`

Albin L. Jones

2.3. **Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann.** Let B be a complete *ccc* Boolean algebra and let τ_s be the topology on B induced by the algebraic convergence of sequences in B.

(1) Either there exists a Maharam submeasure on B or every nonempty open set in (B, τ_s) is topologically dense.
(2) It is consistent that every weakly distributive complete *ccc* Boolean algebra carries a strictly positive Maharam submeasure.
(3) The topological space (B, τ_s) is sequentially compact if and only if the generic extension by B does not add independent reals.

We also give examples of *ccc* forcings adding a real but not independent reals.

This paper seems to extend some of the results announced in Section 2.2 of $\mathcal{SPM}$ **Bulletin** 8.

B. Balcar, T. Jech, and T. Pazák

2.4. **Cardinal invariants $\mathfrak{p}$, $\mathfrak{t}$ and $\mathfrak{h}$ and real functions.** A partial order on a family of continuous functions from a topological space X into $[\omega]^\omega$ is defined as follows

$$f \subseteq^* g \iff f(x) \subseteq^* g(x) \text{ for any } x \in X.$$

For these orders variants of cardinals $\mathfrak{p}$, $\mathfrak{t}$ and $\mathfrak{h}$ are defined and their values are estimated.

Michał Machura

2.5. **A comment on $\mathfrak{p} < \mathfrak{t}$.** We prove that $\mathfrak{p} < \mathfrak{t}$ if, and only if, $({}^\omega\omega, <^*)$ has a peculiar cut.[1] We give a self-contained proof (except using Bell theorem).

`http://arxiv.org/abs/math.LO/0404220`
Saharon Shelah

2.6. **On squares of spaces and F_σ-sets.** We show that the the Continuum Hypothesis implies there exists a Lindelöf space X such that X^2 is the union of two metrizable subspaces but X is not metrizable. This gives a consistent solution to a problem of Balogh, Gruenhage, and Tkachuk. The main lemma is that assuming the the Continuum Hypothesis there exist disjoint sets of reals X and Y such that X is Borel concentrated on Y, i.e., for any Borel set B if $Y \subseteq B$ then $X \setminus B$ is countable, but $X^2 \setminus \Delta$ is relatively F_σ in $X^2 \cup Y^2$.

`http://arxiv.org/abs/math.LO/0404421`
Arnold W. Miller

2.7. **Comparing the uniformity invariants of null sets for different measures.** It is shown to be consistent with set theory that the uniformity invariant for Lebesgue measure is strictly greater than the corresponding invariant for Hausdorff r-dimensional measure where $0 < r < 1$.

`http://arxiv.org/abs/math.LO/0405092`
Saharon Shelah and Juris Steprāns

2.8. **Maximal functions and the additivity of various families of null sets.** It is shown to be consistent with set theory that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ planes in Euclidean 3-space whose union is not null. Similar results are obtained for circles in the plane as well as other geometric objects. The proof relies on results from harmonic analysis about the boundedness of certain maximal operators and a measure theoretic pigeonhole principle.

Juris Steprans

[1]The definition of this "peculiar cut" appears in the paper.

2.9. **How many miles to $\beta\omega$? – Approximating $\beta\omega$ by metric-dependent compactifications.** It is known that the Stone-Čech compactification βX of a non-compact metrizable space X is approximated by the collection of Smirnov compactifications of X for all compatible metrics on X. We investigate the smallest cardinality of a set D of compatible metrics on the countable discrete space ω such that, $\beta\omega$ is approximated by Smirnov compactifications for all metrics in D, but any finite subset of D does not suffice. We also study the corresponding cardinality for Higson compactifications.

`http://arxiv.org/abs/math.GN/0405311`
Masaru Kada, Kazuo Tomoyasu, Yasuo Yoshinobu

2.10. **The cardinal characteristic for relative gamma-sets.** For X a separable metric space define $\mathfrak{p}(X)$ to be the smallest cardinality of a subset Z of X which is not a relative γ-set in X, i.e., there exists an ω-cover of X with no γ-subcover of Z. We give a characterization of $\mathfrak{p}(2^\omega)$ and $\mathfrak{p}(\omega^\omega)$ in terms of definable free filters on ω which is related to the psuedointersection number $\mathfrak{p}$. We show that for every uncountable standard analytic space X, either $\mathfrak{p}(X)=\mathfrak{p}(2^\omega)$ or $\mathfrak{p}(X)=\mathfrak{p}(\omega^\omega)$. We show that the following statements are each relatively consistent with ZFC: (a) $\mathfrak{p}=\mathfrak{p}(\omega^\omega)<\mathfrak{p}(2^\omega)$ and (b) $\mathfrak{p}<\mathfrak{p}(\omega^\omega)=\mathfrak{p}(2^\omega)$.

`http://arxiv.org/abs/math.LO/0405473`
Arnold W. Miller

2.11. **Uncountable intersections of open sets under $\mathbf{CPA}_{\mathrm{prism}}$.** We prove that the Covering Property Axiom $\mathrm{CPA}_{\mathrm{prism}}$, which holds in the iterated perfect set model, implies the following facts.

(1) If G is an intersection of $\aleph_1$-many open sets of a Polish space and G has cardinality continuum, then G contains a perfect set.
(2) There exists a subset G of the Cantor set which is an intersection of $\aleph_1$-many open sets but is not a union of $\aleph_1$-many closed sets.

The example from the second fact refutes a conjecture of Brendle, Larson, and Todorcevic.

`http://www.ams.org/journal-getitem?pii=S0002-9939-04-07475-1`
Krzysztof Ciesielski and Janusz Pawlikowski

2.12. **Covering $\mathbb{R}^{n+1}$ by graphs of n-ary functions and long linear orderings of Turing degrees.** A point $(x_0,\dots,x_n)\in X^{n+1}$ is *covered* by a function $f:X^n\to X$ iff there is a permutation σ of $n+1$ such that $x_{\sigma(0)}=f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

By a theorem of Kuratowski, for every infinite cardinal κ exactly κ n-ary functions are needed to cover all of $(\kappa^{+n})^{n+1}$. We show that for arbitrarily large uncountable κ

it is consistent that the size of the continuum is κ^{+n} and $\mathbb{R}^{n+1}$ is covered by κ n-ary continuous functions.

We study other cardinal invariants of the σ-ideal on $\mathbb{R}^{n+1}$ generated by continuous n-ary functions and finally relate the question of how many continuous functions are necessary to cover $\mathbb{R}^2$ to the least size of a set of parameters such that the Turing degrees relative to this set of parameters are linearly ordered.

http://www.ams.org/journal-getitem?pii=S0002-9939-04-07422-2

Uri Abraham and Stefan Geschke

3. Confrences

3.1. **Foundations of the Formal Sciences V: Infinite Games.** Rheinische Friedrich-Wilhelms-Universität Bonn, Mathematisches Institut, November 26th to 29th, 2004.

http://www.math.uni-bonn.de/people/fotfs/V/

Infinite Games have been investigated by mathematicians since the beginning of the twentieth century and have played a central role in mathematical logic. However, their applications go far beyond mathematics: they feature prominently in theoretical computer science, philosophical Gedankenexperiments, as limit cases in economical applications, and in many other applications. The conference FotFS V wants to bring together researchers from the various areas that employ infinitary game techniques to talk about similarities and dissimilarities of the different approaches and develop cross-cultural bridges.

We invite all researchers from areas applying infinitary game-theoretic methods (economists, mathematicians, logicians, philosophers, computer scientists, sociologists) to submit their papers before September 15th, 2004. Topics will include Games in Algebra and Logic, Games in Higher Set Theory, Games in Set-Theoretic Topology, Infinite Games and Computer Science, Infinite Games in Philosophy, Infinite Evolutionary Games, Machine Games, Game Logics, Infinite Games in the Social Sciences.

Invited Speakers:

- Samson Abramsky, Oxford UK
- Alessandro Andretta, Torino
- Natasha Dobrinen, State College PA
- Ian Hodkinson, London UK
- Kevin Kelly, Pittsburgh PA
- Hamid Sabourian, Cambridge UK
- Marion Scheepers, Boise ID
- Brian Skyrms, Irvine CA

Organizing and Scientific Committee: Stefan Bold (Bonn / Denton TX), Boudewijn de Bruin (Amsterdam), Peter Koepke (Bonn), Benedikt Löwe (Amsterdam / Bonn, Coordinator), Thoralf Räsch (Potsdam), Johan van Benthem (Amsterdam / Stanford).

Coordinating e-mail Address: fotfs@math.uni-bonn.de

Benedikt Loewe

4. Problem of the month

The problem for this issue can be stated with very few definitions. We will state it this way, and then describe the general framework where it arises. Recall that $\mathcal{U}$ is a *large cover* of a set of reals X if each element of X is covered by infinitely many members of $\mathcal{U}$. Let us tentatively say that X has the *splitting property* if each large open cover of X can be split into two disjoint large covers of X. The *Problem of the month* is:

Problem 4.1. *Is it provable that, for all sets of reals X and Y with the splitting property, $X \cup Y$ has the splitting property?*

The following discussion is based on [3]. Assume that $\mathfrak{U}$ and $\mathfrak{V}$ are collections of covers of a space X. The following property was introduced in [2].

$\mathsf{Split}(\mathfrak{U}, \mathfrak{V})$: Every cover $\mathcal{U} \in \mathfrak{U}$ can be split into two disjoint subcovers $\mathcal{V}$ and $\mathcal{W}$ which contain elements of $\mathfrak{V}$.

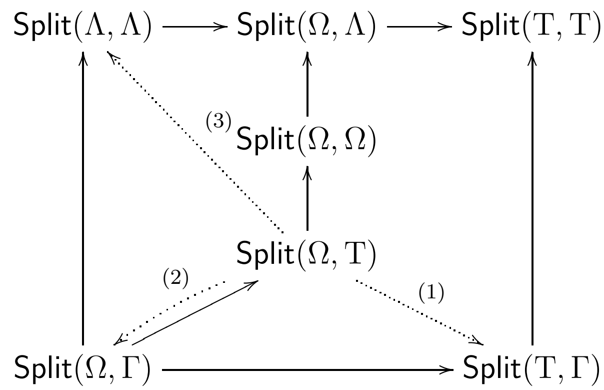
Then the above-mentioned "splitting property" is just $\mathsf{Split}(\Lambda, \Lambda)$, where Λ denotes the collection of all large open covers of the space in question.

Properties of this form are useful in the Ramsey theory of thick covers. The Hurewicz property $\mathsf{U}_{fin}(\Gamma, \Gamma)$ and Rothberger's property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ each implies $\mathsf{Split}(\Lambda, \Lambda)$, and that the Sakai property $\mathsf{S}_1(\Omega, \Omega)$ implies $\mathsf{Split}(\Omega, \Omega)$ [2]. If all finite powers of X have the Hurewicz property (this is equivalent to $\mathsf{S}_{fin}(\Omega^{gp}, \Omega)$), then X satisfies $\mathsf{Split}(\Omega, \Omega)$ [1].

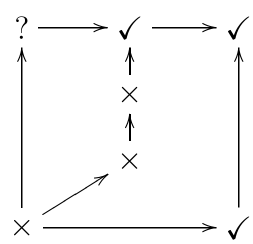
If we consider this prototype with $\mathfrak{U}, \mathfrak{V} \in \{\Lambda, \Omega, \mathrm{T}, \Gamma\}$ we obtain the following 16 properties.

$$
\begin{array}{ccccccc}
\mathsf{Split}(\Lambda,\Lambda) & \longrightarrow & \mathsf{Split}(\Omega,\Lambda) & \longrightarrow & \mathsf{Split}(\mathrm{T},\Lambda) & \longrightarrow & \mathsf{Split}(\Gamma,\Lambda) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathsf{Split}(\Lambda,\Omega) & \longrightarrow & \mathsf{Split}(\Omega,\Omega) & \longrightarrow & \mathsf{Split}(\mathrm{T},\Omega) & \longrightarrow & \mathsf{Split}(\Gamma,\Omega) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathsf{Split}(\Lambda,\mathrm{T}) & \longrightarrow & \mathsf{Split}(\Omega,\mathrm{T}) & \longrightarrow & \mathsf{Split}(\mathrm{T},\mathrm{T}) & \longrightarrow & \mathsf{Split}(\Gamma,\mathrm{T}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathsf{Split}(\Lambda,\Gamma) & \longrightarrow & \mathsf{Split}(\Omega,\Gamma) & \longrightarrow & \mathsf{Split}(\mathrm{T},\Gamma) & \longrightarrow & \mathsf{Split}(\Gamma,\Gamma)
\end{array}
$$

But all properties in the last column are trivial in the sense that all sets of reals satisfy them. On the other hand, all properties but the top one in the first column imply $\binom{\Lambda}{\Omega}$ and are therefore trivial in the sense that no infinite set of reals satisfies any of them. Moreover, the properties $\mathsf{Split}(\mathrm{T},\mathrm{T})$, $\mathsf{Split}(\mathrm{T},\Omega)$, and $\mathsf{Split}(\mathrm{T},\Lambda)$ are equivalent. It is also easy to see that $\mathsf{Split}(\Omega,\Gamma) \Leftrightarrow \binom{\Omega}{\Gamma}$, therefore $\mathsf{Split}(\Omega,\Gamma)$ implies $\mathsf{Split}(\Lambda,\Lambda)$. In [3] it is proved that no implication can be added to the following diagram, except perhaps the dotted ones. (If the dotted implication (1) is true, then so are (2) and (3).)



With regards to the additivity (preservation under taking finite unions) and σ-additivity (countable unions), the following is known (✓means that the property in this position is σ-additive, and $\times$ means that it is not additive).



Thus Problem 4.1, asking whether $\mathsf{Split}(\Lambda,\Lambda)$ is additive, is the only remaining open problem regarding additivity of these properties. In Proposition 1.1 of [3] it is shown that for a set of reals X (in fact, for any hereditarily Lindelöf space X), each large open cover of X contains a *countable* large open cover of X. Consequently, using standard arguments [3], the problem is closely related to the the following one (where $P_\infty(\mathbb{N})$ is the space of all infinite sets of natural numbers, with the topology inherited from $P(\mathbb{N})$, the latter identified with ${}^{\mathbb{N}}\{0,1\}$).

Problem 4.2. *If* R *denotes the sets of reals* X *such that each continuous image of* X *in* $P_\infty(\mathbb{N})$ *is not reaping, then is* R *additive?*

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5. Problems from earlier issues

In this section we list the past problems posed in the $\mathcal{SPM}$ **Bulletin**, in the section *Problem of the month*. For definitions, motivation and related results, consult the corresponding issue.

For conciseness, we make the convention that all spaces in question are zero-dimentional, separable metrizble spaces.

Issue 1. *Is* $\binom{\Omega}{\Gamma} = \binom{\Omega}{\mathrm{T}}$*?*

Issue 2. *Is* $\mathsf{U}_{fin}(\Gamma, \Omega) = \mathsf{S}_{fin}(\Gamma, \Omega)$*? And if not, does* $\mathsf{U}_{fin}(\Gamma, \Gamma)$ *imply* $\mathsf{S}_{fin}(\Gamma, \Omega)$*?*

Issue 3. *Does there exist (in ZFC) a set satisfying* $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$ *but not* $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$*?*

Solution. **Yes** (Lubomyr Zdomsky). □

Issue 4. *Does* $\mathsf{S}_1(\Omega, \mathrm{T})$ *imply* $\mathsf{U}_{fin}(\Gamma, \Gamma)$*?*

Issue 5. *Is* $\mathfrak{p} = \mathfrak{p}^*$*?*

Issue 6. *Does there exist (in ZFC) an uncountable set satisfying* $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$*?*

Issue 7. *Assume that* X *has strong measure zero and* $|X| < \mathfrak{b}$*. Must all finite powers of* X *have strong measure zero?*

Solution. **Yes** (Scheepers; Bartoszyński). □

Issue 8. *Does* $X \notin \mathsf{NON}(\mathcal{M})$ *and* $Y \notin \mathsf{D}$ *imply that* $X \cup Y \notin \mathsf{COF}(\mathcal{M})$*?*

References

[1] Lj. D. R. Kočinac and M. Scheepers, *Combinatorics of open covers (VII): Groupability*, Fundamenta Mathematicae **179** (2003), 131–155.

[2] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology and its Applications **69** (1996), 31–62.

[3] B. Tsaban, *The combinatorics of splittability*, Annals of Pure and Applied Logic, to appear. http://arxiv.org/abs/math.LO/0212312

Previous issues. The first issues of this bulletin, which contain general information (first issue), basic definitions, research announcements, and open problems (all issues) are available online, on `http://arxiv.org/abs/math.GN/`x, where x is 0301011, 0302062, 0303057, 0304087, 0305367, 0312140, and 0401155, 0403369, respectively, for issues number 1 to 8.

Contributions. Please submit your contributions (announcements, discussions, and open problems) by e-mailing us. It is preferred to write them in LaTeX. The authors are urged to use as standard notation as possible, or otherwise give the definitions or a reference to where the notation is explained. Contributions to this bulletin would not require any transfer of copyright, and material presented here can be published elsewhere.

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