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Contents

1. Editor's note

In this issue we announce a fascinating series of works on the comparison of various types of convergence of sequences of functions. Some of these properties are provably related to some of the properties which were introduced in the earlier issues of the $\mathcal{SPM}$ **BULLETIN**, and many problems remain open. Section 2 below, written by Lev Bukovský, contains a brief survey of some of the major open problems in this area.

This issue gives the first example of the importance of the transmission of knowledge between the recipients of this bulletin: One of the announcements implies a solution to one of the problems posed in an *independent* paper announced here (see [§3.7 below]).

The first issues of this bulletin are available online:

(1) First issue: `http://arxiv.org/abs/math.GN/0301011`

(2) Second issue: `http://arxiv.org/abs/math.GN/0302062`

We are looking forward to receive more announcements from other recipients of the bulletin.

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2. Not distinguishing convergences: Open problems

We recall some definitions (see e.g. [§3.3 below]). An open cover $\mathcal{U}$ is *a γ-cover* if every point $x \in X$ is in all but finitely many sets from $\mathcal{U}$. In accordance with W. Hurewicz [11] we define [§3.3 below]:
E*: *for every sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X there exist finite subsets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n\}_{n\in\mathbb{N}}$ is a cover of X.* [1]
$\mathbf{E}^{**}_\omega$: *for every sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of countable open covers of X there exist finite subsets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n\}_{n\in\mathbb{N}}$ is a γ-cover of X or a finite cover of X.* [2]

A countably compact non-compact topological space has property E^{**}_ω and has not property E^*. There are examples of such spaces (e.g. [8], pp. 261–262), however none of them is perfectly normal. The existence of a perfectly normal countably compact non-compact space neither can be proved nor can be refuted in **ZFC** (see e.g. [19]).

Problem 2.1 ([§3.3 below])**.** *Find in ZFC a perfectly normal E^{**}_ω-space which does not possess property* E^*.

We say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ converges *quasi-normally to a function f on X*, (see e.g. [5], in [7] as equally convergent) if there is a sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ (*a control*) converging to 0 such that

$$(\forall x \in X)(\exists n_0)(\forall n \geq n_0)\, |f_n(x) - f(x)| < \varepsilon_n. \tag{1}$$

Similarly, the series $\sum_{n=0}^{\infty} f_n$ converges *pseudo-normally on X* if there is a control sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and (1) holds true (with $f = 0$).

A topological space X is said to be *a wQN-space*, see [6], if from every sequence of continuous functions converging to 0 on X one can choose a quasi-normally convergent subsequence. A topological space X is said to be *a $\Sigma\Sigma^*$-space*, see [§3.1 below], if for every sequence $\{f_n\}_{n\in\mathbb{N}}$ of real functions with non-negative values such that $\sum_{n=0}^{\infty} f_n(x) < \infty$

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[1] This is $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ in Scheepers' terminology adopted in this bulletin (see first issue).

[2] This is $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ in Scheepers' terminology, if $\mathcal{O}$ is restricted to *countable* open covers of X. This restriction can make a difference when X is not Lindelöf.

for every $x \in X$ the series converges also pseudo-normally. Finally a topological space is *a* $\overline{\mathrm{QN}}$*-space*, see [§3.1 below], if every sequence of real functions converging pointwise to a function on X (not necessarily continuous) converges to this function quasi-normally.

In [§3.1 below] the authors show that

$$\Sigma\Sigma^* \to \overline{\mathrm{QN}} \text{ for perfectly normal space.}$$

Usually passing from properties of sequences of real-valued continuous functions to properties of open coverings we need to assume that the considered topological space is perfectly normal. However, both notions $\Sigma\Sigma^*$-space and $\overline{\mathrm{QN}}$-space do not use a notion of an open covering in their definitions. Therefore we suppose that

Problem 2.2. $\Sigma\Sigma^* \to \overline{\mathrm{QN}}$ *for arbitrary topological space.*

For a topological space X and a subset $A \subset X$ we denote

$$\begin{aligned} s_0(A) &= A, \quad s_\xi(A) = \{\lim_{n\to\infty} x_n : x_n \in \bigcup_{\eta<\xi} s_\eta(A) \text{ for each } n \in \mathbb{N}\}, \\ \sigma(A) &= \min\{\xi : s_\xi(A) = s_{\xi+1}(A)\}, \quad \Sigma(X) = \sup\{\sigma(A) : A \subseteq X\}, \end{aligned}$$

The fundamental result in this area is David Fremlin's

Theorem 2.3 ([9])**.** $\Sigma(C_p(X)) = 0,\ 1,\ \omega_1$.

The theorem suggests to define: a topological space X is said to be *an* s_1*-space* if $\Sigma(C_p(X)) = 1$.

In [16] the author introduces the *sequence selection property*, shortly *SSP* of a topological space X: if $\lim_{i\to\infty} f_{n,i}(x) = 0$ for $x \in X$, $n \in \mathbb{N}$, then there are i_n such that $\lim_{n\to\infty} f_{n,i_n}(x) = 0$ for $x \in X$. Actually SSP is equivalent to α_2 property of $C_p(X)$ introduced by A. V. Archangelskij [1].

Theorem 2.4. ([17], implicitly in [9]) SSP $=$ s_1*-space.*

Theorem 2.5. ([17]) SSP $\to$ wQN.

Recently D. Fremlin proved

Theorem 2.6. ([10]) wQN$\to$SPP.

A topological space X is said to be *a* $\mathrm{S}_1(\Gamma,\Gamma)$*-space* if for every sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of γ-covers of X there exists a γ-cover $\{U_n\}_{n\in\mathbb{N}}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$. In [17] the author shows that $\mathsf{S}_1(\Gamma,\Gamma) \to$ s_1-space and conjectured that

Problem 2.7. *Every perfectly normal wQN-space* ($=$ s_1-space) *has property* $\mathsf{S}_1(\Gamma,\Gamma)$.

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3. Research announcements

3.1. **Spaces not distinguishing convergences of real-valued functions.** In [6] we have introduced the notion of a wQN-space as a space in which for every sequence of continuous functions pointwisely converging to 0 there is a subsequence quasi-normally converging to 0. In the present paper we continue this investigation and generalize some concepts touched there. The content is a variety of notions and relationships among them. The result is another scale in the investigation of smallness and the question is how this scale fits with other known scales and whether all relations in it are proper.
The paper appeared in *Topology and its Applications* **112** (2001), 13–40.

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3.2. **Hurewicz Properties, not Distinguishing Convergence Properties and Sequence Selection Properties.** We shall compare several properties of a topological space related to the behavior of open coverings and/or the behavior of sequences of continuous real-valued functions defined on the space. We shall show that there are closed relationships between them and several of them are mutually equivalent.

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3.3. **On Hurewicz Properties.** We investigate Hurewicz properties introduced in [14] and [11] and later introduced related properties of topological spaces. The main result says that for perfectly normal spaces the property mQN introduced in [§3.1 above] is equivalent to Hurewicz property E_ω^{**}. As corollaries we obtain solution of several open problems stated in [§3.1 above]. A complete overview of relationships between the considered properties is presented.

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3.4. **Uncountable $\Sigma\Sigma^*$ subset of $\mathbb{R}$.** If CH holds then there exists an uncountable $X \subset [0,1]$ which belongs to $\Sigma\Sigma^*$.

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3.5. **Spaces not distinguishing convergences.** In the present paper we introduce a convergence condition (Σ') and continue the study of "not distinguish" for various kinds of convergence of sequences of real functions on a topological space started in [6] and [§3.1 above].

We compute cardinal invariants associated with introduced properties of spaces.

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3.6. **A Nonhereditary Borel-cover γ-set.** In this paper we answer some of the questions raised by Bartoszynski and Tsaban [4] concerning hereditary properties of sets defined by certain Borel covering properties:

Theorem. Suppose there is a Borel-cover γ-set [3] of size the continuum. Then there is a Borel-cover γ-set X and subset Y of X which is not even an open-cover γ-set. (In fact there is an of open ω-cover of Y with no τ-subcover.)

It is also shown that CH implies that there exists a Borel-cover γ-set of size ω_1.

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3.7. **Editor's remark: A γ- and σ-set need not be hereditary.** In Problem 7.9 of the announced paper [§3.1 above] it is asked whether every γ-set of reals which is also a σ-set is a *hereditary* γ-set. By [§3.6 above], assuming CH there exists an element of $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ with a subset which is not a γ-set. Clearly $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ implies $\mathsf{S}_1(\Omega, \Gamma)$ (= γ-set), as well as $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. In [18] it is proved that every set satisfying $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is a σ-set. This answers the problem negatively.

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3.8. **The minimal cardinality where the Reznichenko property fails.** According to Reznichenko, a topological space X has the weak Fréchet-Urysohn property if for each subset A of X and each element x in $\overline{A} \setminus A$, there exists a countably infinite pairwise disjoint collection $\mathcal{F}$ of finite subsets of A such that for each neighborhood U of x, $U \cap F \neq \emptyset$ for all but finitely many $F \in \mathcal{F}$. In [13], Kočinac and Scheepers conjecture:

> The minimal cardinality of a set X of real numbers such that $C_p(X)$ does not have the weak Fréchet-Urysohn property is equal to $\mathfrak{b}$.

($\mathfrak{b}$ is the minimal cardinality of an unbounded family in the Baire space ${}^{\mathbb{N}}\mathbb{N}$). We prove the Kočinac-Scheepers conjecture by showing that if $C_p(X)$ has the Reznichenko property, then a continuous image of X cannot be a subbase for a non-feeble filter on $\mathbb{N}$.

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[3] That is, an element of $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.

4. Other announcements

4.1. The Twentyseventh Summer Symposium in Real Analysis. June 23–29, 2003.

During June 23-29, 2003, the Mathematical Institute of Silesian University at Opava will host the Summer Symposium in Real Analysis XXVII. The nature of current work in real analysis is driven by the exchange of ideas generated by real analysts rooted in one subdiscipline of real analysis but with wide ranging interests. This Symposium will highlight lectures by both leading experts and energetic new researchers. Specifically, Summer Symposium XXVII will emphasize recent important work in harmonic analysis, integration theory and a solution of the celebrated Gradient Problem as well as some of the achievements of younger mathematicians in real analysis. In addition, we will provide a vibrant forum for the discussion of research problems, and allot prime speaking time to recent doctoral recipients.

The principal speakers have been invited and at the time of this submission, all have tentatively accepted our invitation.

- Jaroslav Kurzweil (Mathematical Institute of the Academy of Sciences, Prague)
- Zoltan Buczolich (Eotvos Lorand University, Budapest)
- Alexander Olevskii (Tel Aviv University, Israel)

Michigan State University Press will publish the proceedings of Symposium XXVI as a separate volume of the Real Analysis Exchange. Electronic registration for the conference can be found at:

`http://www.math.slu.cz/RealAnalysis/`

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4.2. BEST 2003 (update).

The organizers of the BEST 2003 conference have informed us with the good news that Arnold Miller will attend this conference and give an invited lecture. For more details see [3].

5. Problem of the month

The following problem, which is a variant of Problem 3 in [12], appears as Problem 1 in [§3.3 above] and in [§2 above].

Problem 5.1. *Does there exist (in ZFC) a set $X \subseteq \mathbb{R}$ which has the Menger property $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$ but not the Hurewicz property $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$?*

Assuming the Continuum Hypothesis, one can construct a *Luzin set* $L \subseteq \mathbb{R}$ of size continuum $\mathfrak{c}$, that is, such that for each meager (=first category) set M, $L \cap M$ is countable. Such a set L is *concentrated* on each of its countable dense subsets D (that is, for each open set $U \supseteq D$, $L \setminus U$ is countable), and therefore has Rothberger's property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$, which implies Menger's property $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ (see, e.g., [15]). On the other hand, in [12] it is proved that every set with the Hurewicz property is (perfectly) meager. Thus, assuming the Continuum Hypothesis, the answer to the above problem is negative. But the Continuum Hypothesis is not necessary to get a negative answer: Let $\mathcal{M}$ denote the collection of meager sets of reals, and write

$$\begin{aligned}\mathsf{cov}(\mathcal{M}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{M} \text{ and } \cup\mathcal{F} = \mathbb{R}\}\\ \mathsf{cof}(\mathcal{M}) &= \min\{|\mathcal{F}| : (\forall M \in \mathcal{M})(\exists F \in \mathcal{F})\ M \subseteq F\}\end{aligned}$$

In [18] it is shown that it is enough to assume that $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$ (this hypothesis is strictly weaker than the Continuum Hypothesis [2]) for the above arguments to work (with some necessary modifications). The Problem of the Month asks whether the assumption $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$ can be completely removed.

The papers [12], [17], and [4] deal with constructions in ZFC of sets of reals with the Hurewicz property, and seem to be relevant to the problem.

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