

SPM BULLETIN

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CONTENTS

1. Editor's note	1
2. Research announcements	2
2.1. On two problems of Erdős and Hechler: New methods in singular Madness	2
2.2. Almost isometric embeddings of metric spaces	2
2.3. Half of an inseparable pair	2
2.4. Covering the Baire space by families which are not finitely dominating	2
2.5. Compact Scattered Spaces in Forcing Extensions	3
2.6. A note on an example by van Mill	3
2.7. Spaces on which every pointwise convergent series of continuous functions converges pseudo-normally	3
2.8. A semifilter approach to selection principles	3
2.9. The combinatorics of τ -covers	4
3. The combinatorics of τ -covers: 4 + 21 problems solved	4
3.1. A modified table of open problems	6
4. Problem of the month	7
5. Problems from earlier issues	7
References	8

1. EDITOR'S NOTE

We are glad to announce the solution of $4 + 21 + \frac{1}{2}$ (!) problems posed in earlier issues of the *SPM Bulletin*; the " $\frac{1}{2}$ " standing for a "consistently yes" answer of Zdomsky to the last issue's *Problem of the month*.

Contributions to the next issue are, as always, welcome.

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2. RESEARCH ANNOUNCEMENTS

2.1. On two problems of Erdős and Hechler: New methods in singular Madness. For an infinite cardinal μ , $\text{MAD}(\mu)$ denotes the set of all cardinalities of *nontrivial maximal almost disjoint families* over μ .

Erdős and Hechler proved in [1] the consistency of $\mu \in \text{MAD}(\mu)$ for a singular cardinal μ and asked if it was ever possible for a singular μ that $\mu \notin \text{MAD}(\mu)$, and also whether $2^{\text{cf}\mu} < \mu \implies \mu \in \text{MAD}(\mu)$ for every singular cardinal μ .

We introduce a new method for controlling $\text{MAD}(\mu)$ for a singular μ and, among other new results about the structure of $\text{MAD}(\mu)$ for singular μ , settle both problems affirmatively.

<http://www.ams.org/journal-getitem?pii=S0002-9939-04-07580-X>
Menahem Kojman, Wieslaw Kubis, and Saharon Shelah

2.2. Almost isometric embeddings of metric spaces. We investigate a relations of almost isometric embedding and almost isometry between metric spaces and prove that with respect to these relations:

- (1) There is a countable universal metric space.
- (2) There may exist fewer than continuum separable metric spaces on \aleph_1 so that every separable metric space is almost isometrically embedded into one of them when the continuum hypothesis fails.
- (3) There is no collection of fewer than continuum metric spaces of cardinality \aleph_2 so that every ultra-metric space of cardinality \aleph_2 is almost isometrically embedded into one of them if $\aleph_2 < 2^{\aleph_0}$.

We also prove that various spaces X satisfy that if a space Y is almost isometric to X than Y is isometric to X .

<http://arxiv.org/abs/math.LO/0406530>
Menachem Kojman and Saharon Shelah

2.3. Half of an inseparable pair. A classical theorem of Luzin is that the separation principle holds for the Π_α^0 sets but fails for the Σ_α^0 sets. We show that for every Σ_α^0 set A which is not Π_α^0 there exists a Σ_α^0 set B which is disjoint from A but cannot be separated from A by a Δ_α^0 set C . Assuming Π_1^1 -determinacy it follows from a theorem of Steel that a similar result holds for Π_1^1 sets. On the other hand assuming $V = L$ there is a proper Π_1^1 set which is not half of a Borel inseparable pair. These results answer questions raised by F. Dashiell.

<http://arxiv.org/abs/math.LO/0407405>
Arnold W. Miller

2.4. Covering the Baire space by families which are not finitely dominating. It is consistent (relative to ZFC) that each union of $\max\{\mathfrak{b}, \mathfrak{g}\}$ many families in the Baire space which are not finitely dominating is not dominating. In particular:

- (1) It is consistent that each union of $\max\{\mathfrak{b}, \mathfrak{g}\}$ many sets with the Scheepers property $U_{fin}(\mathcal{O}, \Omega)$ has Menger's property $U_{fin}(\mathcal{O}, \mathcal{O})$.
- (2) It is consistent that for each nonprincipal ultrafilter \mathcal{U} , the cofinality of the reduced ultrapower ${}^{\mathbb{N}}\mathbb{N}/\mathcal{U}$ is greater than $\max\{\mathfrak{b}, \mathfrak{g}\}$.

The model is constructed by oracle chain condition forcing, to which we give a self-contained introduction.

<http://arxiv.org/abs/math.LO/0407487>

Heike Mildenberger, Saharon Shelah, and Boaz Tsaban

2.5. Compact Scattered Spaces in Forcing Extensions. We consider the cardinal sequences of compact scattered spaces in models where the Continuum Hypothesis is false. We describe a number of models where the continuum is \aleph_2 in which no such space can have \aleph_2 countable levels.

<http://arxiv.org/abs/math.GN/0408039>

Kenneth Kunen

2.6. A note on an example by van Mill. Improving on an earlier example by J. van Mill, we prove that there exists a zero-dimensional compact space of countable π -weight and uncountable character which is homogeneous under $MA + \neg CH$, but not under CH .

<http://arxiv.org/abs/math.GN/0408134>

K. P. Hart and G. J. Ridderbos

2.7. Spaces on which every pointwise convergent series of continuous functions converges pseudo-normally. A topological space X is a $\Sigma\Sigma^*$ -space provided that, for every sequence $\langle f_n \rangle_{n=0}^{\infty}$ of continuous functions from X to \mathbb{R} , if the series $\sum_{n=0}^{\infty} |f_n|$ converges pointwise, then it converges pseudo-normally. We show that every regular Lindelöf $\Sigma\Sigma^*$ -space has the Rothberger property. We also construct, under the continuum hypothesis, a $\Sigma\Sigma^*$ -subset of \mathbb{R} of cardinality continuum.

<http://www.ams.org/journal-getitem?pii=S0002-9939-04-07376-9>

Lev Bukovsky and Krzysztof Ciesielski

2.8. A semifilter approach to selection principles. In this paper we develop the semifilter approach to the classical Menger and Hurewicz properties and show that the small cardinal \mathfrak{g} is a lower bound of the additivity number of the σ -ideal generated by Menger subspaces of the Baire space, and under $\mathfrak{u} < \mathfrak{g}$ every subset X of the real line with the property $\text{Split}(\Lambda, \Lambda)$ is Hurewicz, and thus it is consistent with ZFC that the property $\text{Split}(\Lambda, \Lambda)$ is preserved by unions of less than \mathfrak{b} subsets of the real line.

Lubomyr Zdomsky

2.9. The combinatorics of τ -covers. We solve four out of the six open problems concerning critical cardinalities of topological diagonalization properties involving τ -covers, show that the remaining two cardinals are equal, and give a consistency result concerning this remaining cardinal. Consequently, 21 open problems concerning potential implications between these properties are settled. We also give structural results based on the combinatorial techniques.

For further details see [§3 below].

<http://arxiv.org/abs/math.GN/0409068>

Heike Mildenberger, Saharon Shelah, and Boaz Tsaban

3. THE COMBINATORICS OF τ -COVERS: 4 + 21 PROBLEMS SOLVED

In Section 4 of the fourth issue of the *SPM Bulletin*, a discussion is made of the Scheepers diagram of selection hypotheses, when extended by allowing τ -covers. 6 critical cardinalities in that diagram were unknown. Moreover, 76 potential implications among properties in this diagram were unsettled.

These remaining cardinalities are addressed in [§2.8 above]. We give here one example of that treatment, and quote the main results. Everything in the remainder of this subsection is quoted, without further notice, from the paper [§2.8 above].

Let C_Γ , C_T , and C_Ω denote the collections of *clopen* γ -covers, τ -covers, and ω -covers of X , respectively. Recall that, since we are dealing with sets of reals, we may assume that all open covers are countable. Restricting attention to countable covers, we have the following, where an arrow denotes inclusion:

$$\begin{array}{ccccccc} \mathcal{B}_\Gamma & \rightarrow & \mathcal{B}_T & \rightarrow & \mathcal{B}_\Omega & \rightarrow & \mathcal{B} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \Gamma & \rightarrow & T & \rightarrow & \Omega & \rightarrow & \mathcal{O} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ C_\Gamma & \rightarrow & C_T & \rightarrow & C_\Omega & \rightarrow & C \end{array}$$

As each of the properties $\Pi(\cdot, \cdot)$, $\Pi \in \{\mathbf{S}_1, \mathbf{S}_{fin}, \mathbf{U}_{fin}\}$, is monotonic in its first variable, we have that for each $x, y \in \{\Gamma, T, \Omega, \mathcal{O}\}$,

$$\Pi(\mathcal{B}_x, \mathcal{B}_y) \rightarrow \Pi(x, y) \rightarrow \Pi(C_x, C_y)$$

(here $C_\mathcal{O} := C$ and $\mathcal{B}_\mathcal{O} := \mathcal{B}$). Consequently,

$$\text{non}(\Pi(\mathcal{B}_x, \mathcal{B}_y)) \leq \text{non}(\Pi(x, y)) \leq \text{non}(\Pi(C_x, C_y)).$$

Definition 3.1. We use the short notation \forall^∞ for “for all but finitely many” and \exists^∞ for “there exist infinitely many”.

- (1) $A \in \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ is a γ -array if $(\forall n)(\forall^\infty m) A(n, m) = 1$.
- (2) $\mathcal{A} \subseteq \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ is a γ -family if each $A \in \mathcal{A}$ is a γ -array.
- (3) A family $\mathcal{A} \subseteq \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ is *finitely τ -diagonalizable* if there exist finite (possibly empty) subsets $F_n \subseteq \mathbb{N}$, $n \in \mathbb{N}$, such that:

- (a) For each $A \in \mathcal{A}$: $(\exists^\infty n)(\exists m \in F_n) A(n, m) = 1$;
- (b) For each $A, B \in \mathcal{A}$:
 - Either $(\forall^\infty n)(\forall m \in F_n) A(n, m) \leq B(n, m)$,
 - or $(\forall^\infty n)(\forall m \in F_n) B(n, m) \leq A(n, m)$.

Translating the notions of covers into corresponding combinatorial notions, one obtains the following. (Notice that $\{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ is topologically the same as the Cantor space ${}^{\mathbb{N}}\{0, 1\}$.)

Theorem 3.2. *For a set of reals X , the following are equivalent:*

- (1) X satisfies $S_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_T)$; and
- (2) For each Borel function $\Psi : X \rightarrow \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$, if $\Psi[X]$ is a γ -family, then it is finitely τ -diagonalizable.

The corresponding assertion for $S_{fin}(C_\Gamma, C_T)$ holds when “Borel” is replaced by “continuous”.

Then the following is proved.

Lemma 3.3. *The minimal cardinality of a γ -family which is not finitely τ -diagonalizable is \mathfrak{b} .*

Having Theorem 3.2 and Lemma 3.3, we get that

$$\mathfrak{b} = \text{non}(S_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_T)) \leq \text{non}(S_{fin}(\Gamma, T)) \leq \text{non}(S_{fin}(C_\Gamma, C_T)) = \mathfrak{b},$$

and therefore $\text{non}(S_{fin}(\Gamma, T)) = \mathfrak{b}$. It follows that $\text{non}(S_1(\Gamma, T)) = \mathfrak{b}$, and using a similar approach, it is proved that $\text{non}(S_1(T, T)) = \mathfrak{t}$, and $\text{non}(S_{fin}(T, T)) = \min\{\mathfrak{s}, \mathfrak{b}\}$.

It is not difficult to see that $\text{non}(S_{fin}(T, \Omega)) = \text{non}(S_{fin}(T, \mathcal{O}))$, call this joint cardinal \mathfrak{o} , the *o*-diagonalization number; the reason for this to be explained soon. The surviving properties (in the open case) appear in Figure 1, with their critical cardinalities, and serial numbers (for later reference). The newly found cardinalities are framed.

By Figure 1,

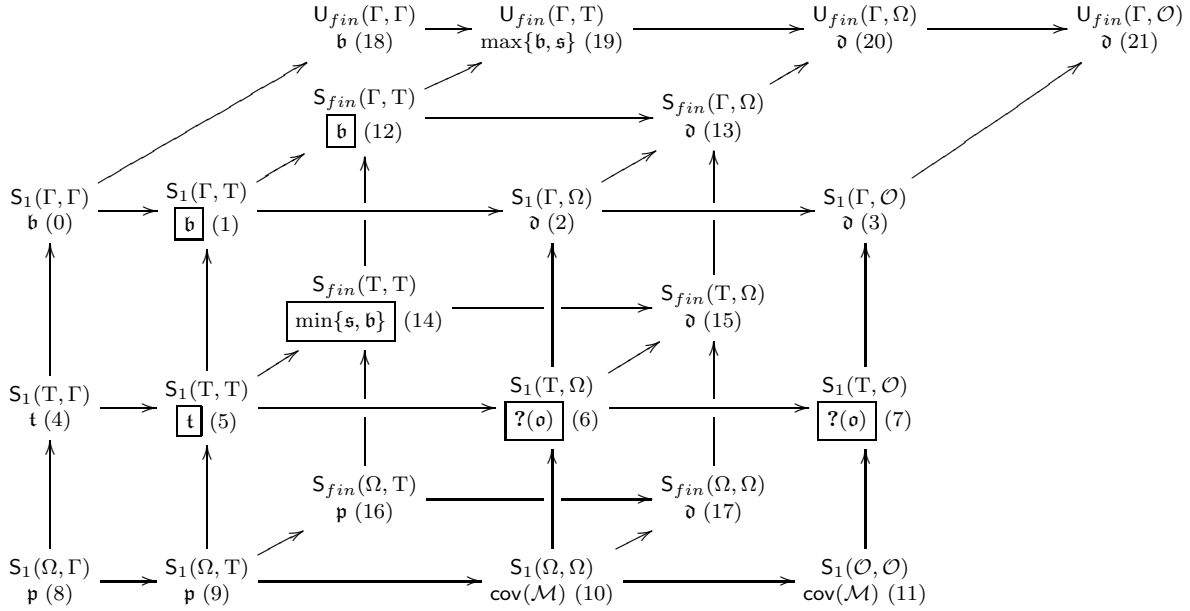
$$\text{cov}(\mathcal{M}) = \text{non}(S_1(\mathcal{O}, \mathcal{O})) \leq \text{non}(S_1(T, \mathcal{O})) \leq \text{non}(S_1(\Gamma, \mathcal{O})) = \mathfrak{d},$$

thus $\text{cov}(\mathcal{M}) \leq \mathfrak{o} \leq \mathfrak{d}$.

Definition 3.4. A τ -family \mathcal{A} is *o*-diagonalizable if there exists a function $g : \mathbb{N} \rightarrow \mathbb{N}$, such that:

$$(\forall A \in \mathcal{A})(\exists n) A(n, g(n)) = 1.$$

As in Theorem 3.2, $S_1(\mathcal{B}_T, \mathcal{B})$ and $S_1(C_T, C)$ have a natural combinatorial characterization. This characterization implies that \mathfrak{o} is equal to the minimal cardinality of a τ -family that is not *o*-diagonalizable. A detailed study of \mathfrak{o} is initiated; the main results being that consistently $\mathfrak{o} < \min\{\mathfrak{h}, \mathfrak{s}, \mathfrak{b}\}$, and that in many standard models of set theory (Cohen, Random, Hechler, Laver, Mathias, and Miller), $\text{cov}(\mathcal{M}) = \mathfrak{o}$.

FIGURE 1. The Scheepers diagram, enhanced with τ -covers

Problem 3.5 (Problem 4.13 in [§2.8 above]). *Is $\text{cov}(\mathcal{M}) = \mathfrak{o}$?*

It is worthwhile mentioning that this problem, which originated from the topological studies of the minimal tower problem, is of similar flavor: It is well-known that if $\mathfrak{p} = \aleph_1$, then $\mathfrak{t} = \aleph_1$ too. We have a similar assertion for $\text{cov}(\mathcal{M})$ and \mathfrak{o} : If $\text{cov}(\mathcal{M}) = \aleph_1 < \mathfrak{b}$, then $\text{cov}(\mathcal{M}) = \mathfrak{o}$.

3.1. A modified table of open problems. It is possible that the diagram in Figure 1 is incomplete: There are many unsettled possible implications in it. After [3, 2], there remained 76 potential implications which were not proved or ruled out. The mentioned results of [§2.8 above] rule out 21 of these implications (the reasoning is as follows: If P and Q are properties with $\text{non}(P) < \text{non}(Q)$ consistent, then Q does not imply P), so that 55 implications remain unsettled. The situation is summarized in Table 1, which updates the corresponding table given in Issue 4 of the *SPM Bulletin*.

Each entry (i, j) (i th row, j th column) contains a symbol. \checkmark means that property (i) in Figure 1 implies property (j) in Figure 1. \times means that property (i) does not (provably) imply property (j) , and $?$ means that the corresponding implication is still unsettled. The new results are framed.

Problem 3.6. *Settle any of the remaining 55 implication in Table 1.*

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	×	?	×	×	✓	✓	✓	✓
1	?	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	×	?	×	×	?	✓	✓	✓
2	×	×	✓	✓	×	×	×	×	×	×	×	×	×	✓	×	?	×	×	×	×	✓	✓
3	×	×	×	✓	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	✓
4	✓	✓	✓	✓	?	✓	✓	✓	×	×	?	?	✓	✓	✓	✓	×	?	✓	✓	✓	✓
5	?	✓	✓	✓	?	✓	✓	✓	×	×	?	?	✓	✓	✓	✓	×	?	?	✓	✓	✓
6	×	×	✓	✓	×	×	✓	✓	×	×	?	?	×	✓	×	✓	×	?	×	×	✓	✓
7	×	×	×	✓	×	×	×	✓	×	×	×	?	×	×	×	×	×	×	×	×	×	✓
8	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
9	?	✓	✓	✓	?	✓	✓	✓	?	✓	✓	✓	✓	✓	✓	✓	✓	✓	?	✓	✓	✓
10	×	×	×	✓	×	×	✓	✓	×	×	✓	×	×	✓	×	✓	×	✓	×	×	✓	✓
11	×	×	×	✓	×	×	×	✓	×	×	×	×	✓	×	×	×	×	×	×	×	×	✓
12	?	?	?	?	×	×	×	×	×	×	×	×	✓	✓	×	?	×	×	?	✓	✓	✓
13	×	×	×	×	×	×	×	×	×	×	×	×	×	✓	×	?	×	×	×	×	✓	✓
14	?	?	?	?	×	×	×	×	×	×	×	×	✓	✓	✓	✓	×	?	?	✓	✓	✓
15	×	×	×	×	×	×	×	×	×	×	×	×	×	✓	×	✓	×	?	×	×	✓	✓
16	?	?	?	?	?	?	?	?	?	?	?	?	✓	✓	✓	✓	✓	✓	?	✓	✓	✓
17	×	×	×	×	×	×	×	×	×	×	×	×	×	✓	×	✓	×	✓	×	×	✓	✓
18	×	×	×	×	×	×	×	×	×	×	×	×	×	?	×	?	×	×	✓	✓	✓	✓
19	×	×	×	×	×	×	×	×	×	×	×	×	×	?	×	?	×	×	✓	✓	✓	✓
20	×	×	×	×	×	×	×	×	×	×	×	×	×	?	×	?	×	×	×	×	✓	✓
21	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	✓

TABLE 1. Known implications and nonimplications

4. PROBLEM OF THE MONTH

See Problem 3.5 above.

5. PROBLEMS FROM EARLIER ISSUES

In this section we list the past problems posed in the *SPM Bulletin*, in the section *Problem of the month*. For definitions, motivation and related results, consult the corresponding issue.

For conciseness, we make the convention that all spaces in question are zero-dimensional, separable metrizable spaces.

Issue 1. $Is \left(\begin{smallmatrix} \Omega \\ \Gamma \end{smallmatrix}\right) = \left(\begin{smallmatrix} \Omega \\ T \end{smallmatrix}\right) ?$

Issue 2. $Is U_{fin}(\Gamma, \Omega) = S_{fin}(\Gamma, \Omega) ?$ And if not, does $U_{fin}(\Gamma, \Gamma)$ imply $S_{fin}(\Gamma, \Omega) ?$

Issue 3. Does there exist (in ZFC) a set satisfying $U_{fin}(\mathcal{O}, \mathcal{O})$ but not $U_{fin}(\mathcal{O}, \Gamma) ?$

Solution. **Yes** (Lubomyr Zdomsky). □

Issue 4. Does $S_1(\Omega, T)$ imply $U_{fin}(\Gamma, \Gamma) ?$

Issue 5. $Is \mathfrak{p} = \mathfrak{p}^* ?$

Issue 6. Does there exist (in ZFC) an uncountable set satisfying $S_1(\mathcal{B}_\Gamma, \mathcal{B}) ?$

Issue 7. Assume that X has strong measure zero and $|X| < \mathfrak{b}$. Must all finite powers of X have strong measure zero?

Solution. **Yes** (Scheepers; Bartoszyński). □

Issue 8. Does $X \notin \text{NON}(\mathcal{M})$ and $Y \notin \text{D}$ imply that $X \cup Y \notin \text{COF}(\mathcal{M})$?

Issue 9. Is $\text{Split}(\Lambda, \Lambda)$ preserved under taking finite unions?

Solution. Consistently yes (Zdomsky [§2.8 above]). We conjecture that it is also consistently no. (This should hold under CH.) □

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- [2] S. Shelah and B. Tsaban, *Critical cardinalities and additivity properties of combinatorial notions of smallness*, Journal of Applied Analysis **9** (2003), 149–162. <http://arxiv.org/abs/math.LO/0304019>
- [3] B. Tsaban, *Selection principles and the minimal tower problem*, Note di Matematica, to appear. <http://arxiv.org/abs/math.LO/0105045>

Previous issues. The first issues of this bulletin, which contain general information (first issue), basic definitions, research announcements, and open problems (all issues) are available online, on <http://arxiv.org/abs/math.GN/x>, where x is 0301011, 0302062, 0303057, 0304087, 0305367, 0312140, and 0401155, 0403369, 0406411, respectively, for issues number 1 to 9.

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