Complexity & Characterization of Set Splitting

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Outline

Background

p-Splittability

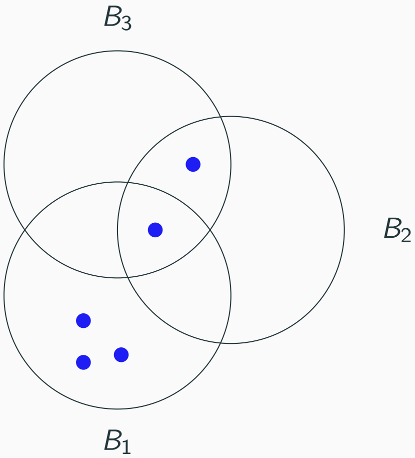
Let $\mathcal{B} = \{B_1, \ldots, B_n\} \subseteq \mathcal{P}(X)$ be a collection of subsets of a finite set X, and fix $0 < p < 1$.

Definition

$\mathcal{B}$ is ***p*-splittable** if there exists a set $S \subseteq X$ such that

$$|B_i \cap S| = \lfloor p|B_i| \rceil .$$

Example $\left(p = \frac{1}{2}\right)$



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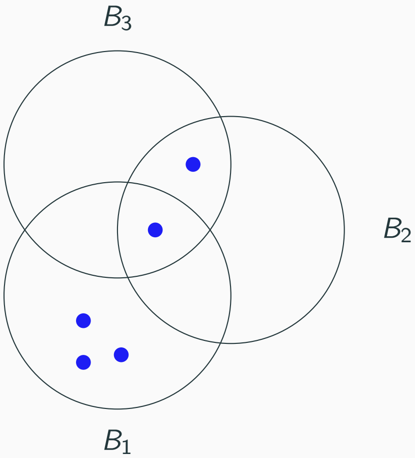

Want:

$$|B_1 \cap S| = 2 = \left\lfloor \frac{1}{2} \cdot 4 \right\rfloor$$
$$|B_2 \cap S| = 1 = \left\lfloor \frac{1}{2} \cdot 2 \right\rfloor$$
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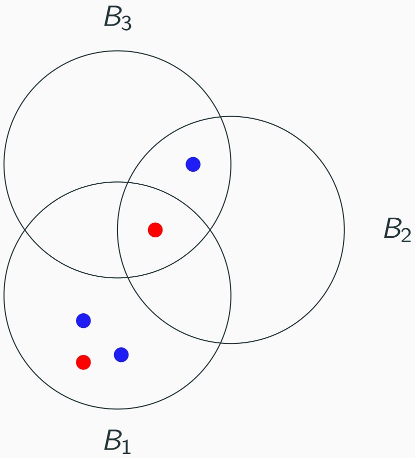

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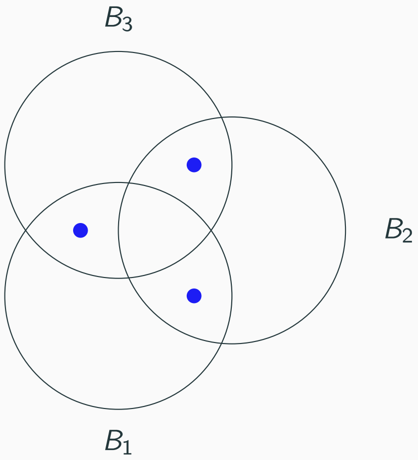

Discrepancy

As before, let $\mathcal{B} = \{B_1, \ldots, B_n\} \subseteq \mathcal{P}(X)$ be a collection of subsets of a finite set X.

Definition

The **discrepancy of** $\mathcal{B}$ is

$$\operatorname{disc}(\mathcal{B}) := \min_{S \subseteq X} \max_{B_i \in \mathcal{B}} \big| |B_i \cap S| - |B_i \setminus S| \big|$$

Bounding Discrepancy

Theorem (Spencer 1985)

An n-set collection $\mathcal{B}$ *has* $\operatorname{disc}(\mathcal{B}) \leq K\sqrt{n}$*, with* K *an absolute constant.*

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- It is possible to efficiently find a set S to witness the bound above.
- However, a collection $\mathcal{B}$ may have $\mathrm{disc}(\mathcal{B}) \ll K\sqrt{n}$. Can you efficiently find a witness for this fact as well?

Discrepancy & Splittability, Together in Concert

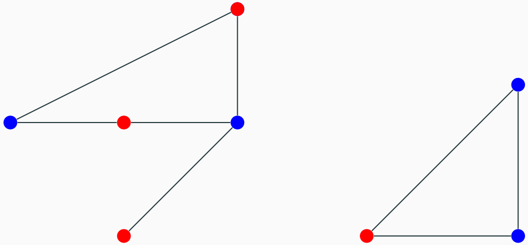
A collection $\mathcal{B}$ is $\frac{1}{2}$-splittable if and only if $\mathrm{disc}(\mathcal{B}) \leq 1 \ll K\sqrt{n}$.

Question

Is there an efficient algorithm to decide whether $\mathrm{disc}(\mathcal{B}) \leq 1$ and find a witness when this is the case?

Hypergraph coloring

We call a hypergraph **2-colorable** if there is a 2-coloring of its vertices such that no hyperedge contains only vertices of a single color.



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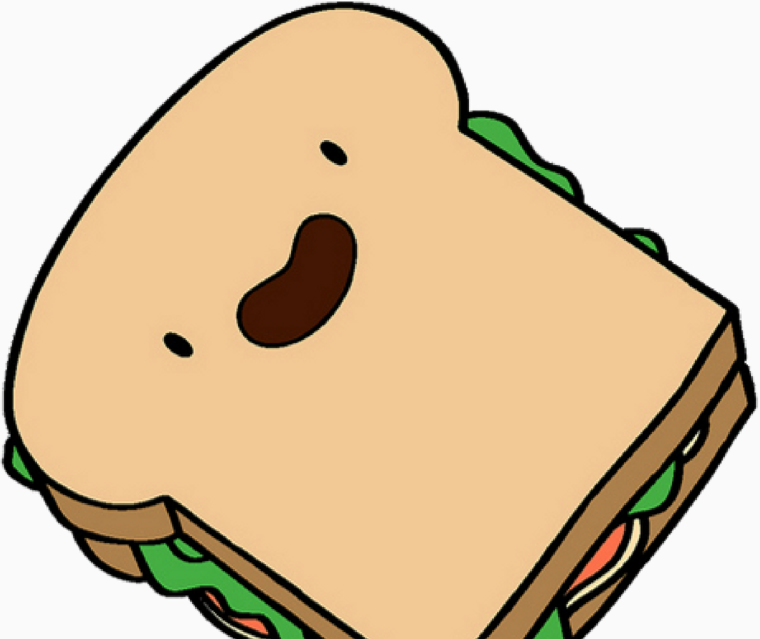
Question

Can we also determine $\frac{1}{2}$-splittability for hypergraphs efficiently?

Ham Sandwiches



Ham Sandwiches

Let $B_1, \ldots, B_n$ be finite sets of points in $\mathbb{R}^n$.

Theorem (Discrete Ham Sandwich Theorem)

There exists a hyperplane H such that for all $i \in [n]$, $|H^+ \cap B_i| \leq \frac{|B_i|}{2}$ and $|H^- \cap B_i| \leq \frac{|B_i|}{2}$.

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Question

When can we bisect the B_i as above, but with an H such that $H \cap B_i = \emptyset$ for $i \in [n]$?

Characterization of Splittable Collections

$\frac{1}{2}$-Splitting for 2-set Collections

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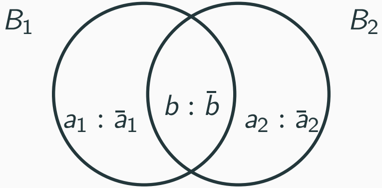

Want $\bar{a}_1 + \bar{b} = \lfloor \frac{a+b}{2} \rceil$ and $\bar{b} + \bar{a}_2 = \lfloor \frac{b+c}{2} \rceil$.

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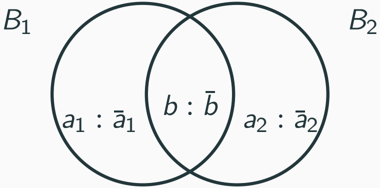

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Let $\bar{a}_1 = \left\lfloor \frac{a}{2} \right\rfloor$, $\bar{b} = \left\lceil \frac{b}{2} \right\rceil$, and $\bar{a}_2 = \left\lfloor \frac{c}{2} \right\rfloor$, satisfying the conditions.

$\frac{1}{2}$-Splitting for 3-set Collections

Theorem

A collection of three sets is not splittable if and only if both

- *Only its bi-fold regions are nonempty*
- *All of its bi-fold Venn regions have odd size.*

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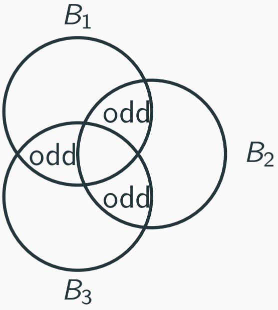

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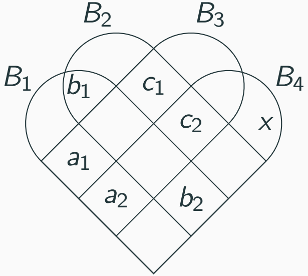
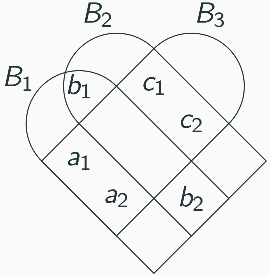
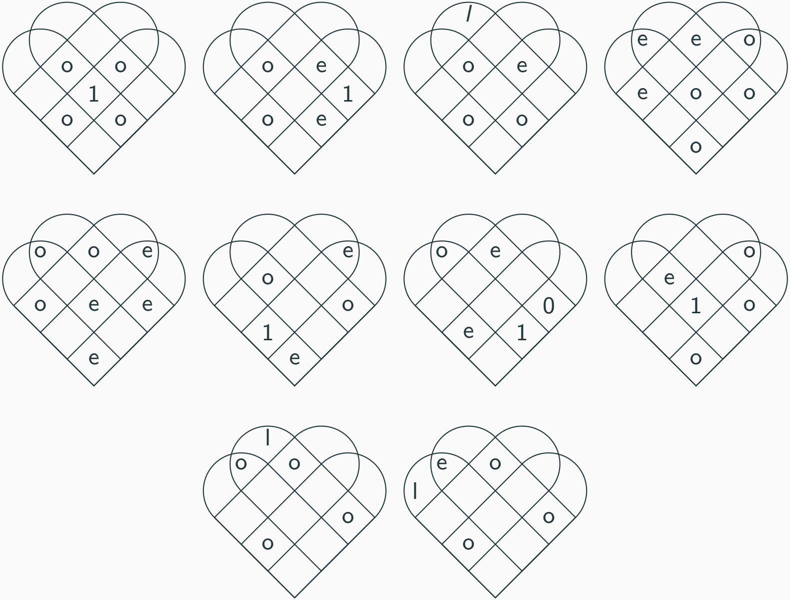
Pattern 1: an unsplittable 3-set configuration within a 4 set configuration.





Unsplittable Configurations for 4-set Collections



Reduction Lemma for $\frac{1}{2}$-Splitting

To prove this theorem:

- Check all cases with region size from 0 to 3 on a supercomputer
- Manually sort the output
- Generalize the conclusion using the lemma below.

Reduction Lemma for $\frac{1}{2}$-Splitting

Lemma

If $\mathcal{B}$ is splittable, then $\mathcal{B}$ remains splittable when an even number of elements are added to any of its regions.

- Take the original splitter and add half of the new added elements.

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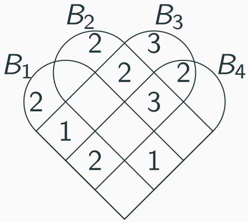
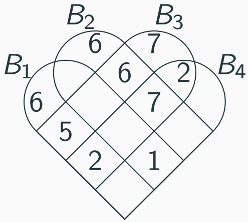
- Given a 4-set collection, we can mod the region size by 4.





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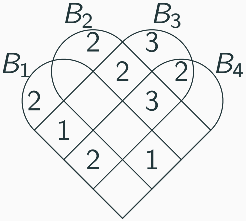

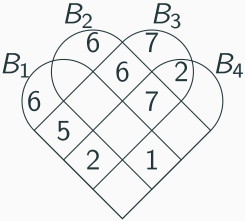

- Although the converse does not hold, this lemma reduces the number of cases significantly and is very helpful to characterize large collections.

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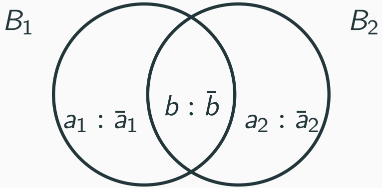


- WLOG, assume $p \leq \frac{1}{2}$.
- It suffices to find integers $\bar{a}_i$ and b such that (i) $0 \leq \bar{a}_i \leq a_i$, (ii) $0 \leq \bar{b} \leq b$, and (iii) $\bar{a}_i + \bar{b} = \lfloor p(a_i + b) \rceil = t_i$.

p-Splitting for 2-set Collections

- Let $\bar{b} = \lfloor pb \rceil$ and $\bar{a}_i = t_i - \bar{b}$ so that (ii) and (iii) are clearly satisfied.
- Left to show that $0 \leq \bar{a}_i = t_i - \bar{b} \leq a_i$
- Let $\varepsilon = \bar{b} - pb$ for the rounding error in computing $\bar{b}$
- Let $\epsilon_i = t_i - p(a_i + b)$ for the rounding error in computing t_i.
- Then, we have

$$\bar{a}_i = t_i - \bar{b} = pa_i + (\epsilon_i - \varepsilon).$$

Since $|\epsilon_i| \leq \frac{1}{2}$ and $|\varepsilon| \leq \frac{1}{2}$ we know that $|\epsilon_i - \varepsilon| \leq 1$. Assuming $a_i > 0$ the above equation gives $-1 < \bar{a}_i < a_i + 1$, and since $\bar{a}_i$ and a_i are integers, (i) is satisfied. On the other hand if $a_i = 0$ then $\bar{a}_i = 0$ too and (i) is clearly satisfied.

p-Splitting for 3-set Collections

Lemma

Assume that $p \leq \frac{1}{2}$, and let $\mathcal{B} = \{B_1, B_2, B_3\}$ be given. Assume all Venn regions of $\mathcal{B}$ of multiplicity 1 are empty. Then $\mathcal{B}$ is p-unsplittable if and only if $\sum t_i$ is odd and one of the conditions holds:

1. $c = 0$; *or*
2. $pc \leq \frac{1}{2}$ *and some* $2pb_i + pc - 1 + \rho_i < 0$.

p-Splitting for 3-set Collections

Lemma

Assume that $p \leq \frac{1}{2}$*, and let* $\mathcal{B} = \{B_1, B_2, B_3\}$ *be given. Let* $\mathcal{B}' = \{B'_1, B'_2, B'_3\}$ *be the corresponding collection with all elements of multiplicity* 1 *deleted. Then* $\mathcal{B}$ *is p-splittable if and only if either:*

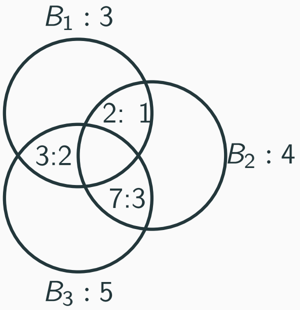
1. $\mathcal{B}'$ *is p-splittable; or*
2. $\mathcal{B}'$ *is p-unsplittable and at least one of the* a_i *is sufficiently large.*

Parity Lemma

Lemma (Parity Lemma)

If a collection of n sets has only its $k\phi$-fold regions nonempty for $k \geq 1$ and some $1 \leq \phi \leq n$, and the set targets $t_1, \dots t_n$ (not necessarily p-splitting targets) are achievable, then the sum of the targets is divisible by ϕ.

- Each element in the splitter appears $k\phi$ times in the sum of the targets.



Partial Converse to Parity Lemma

The converse holds when the collection only uses $n-1$-fold regions.

Lemma

Let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be a collection of sets with only its $n-1$-fold regions nonempty. Then $\mathcal{B}$ is p-splittable if and only if and $\sum_{i=1}^{n} [p|B_i|]$ is divisible by $n-1$.

When n gets large, things get more complex. It is very difficult to find a general rule for determining p-splittability.

Complexity of p-Splitting

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- It is easily seen that the p-splitting problem is in NP, as if I propose a splitter to the collection, one could check that every set of the collection has half of the elements in the splitter.

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- To prove that a problem is NP complete, you must reduce some problem that is known to be NP complete to the problem you are considering.
- Examples of NP complete problems include the Traveling Salesman Problem, the Knapsack Problem, and the Zero-One Equations (ZOE).

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- Example:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So a solution to the previous ZOE would be $\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$

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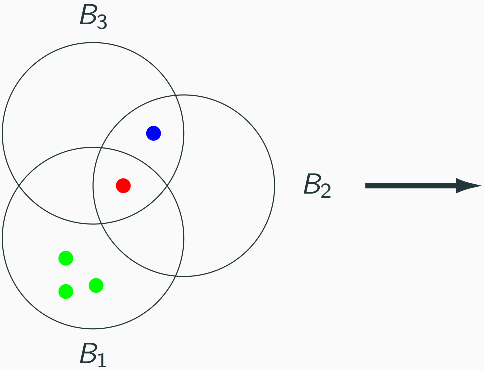

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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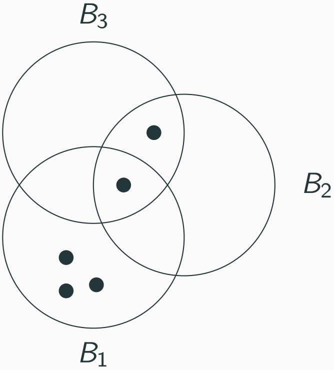

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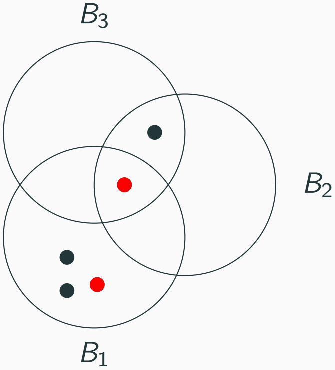

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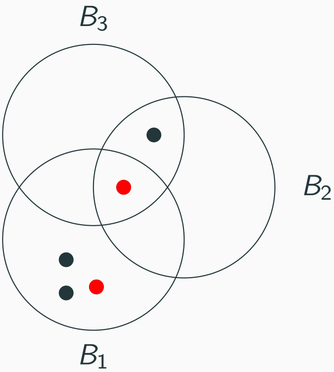

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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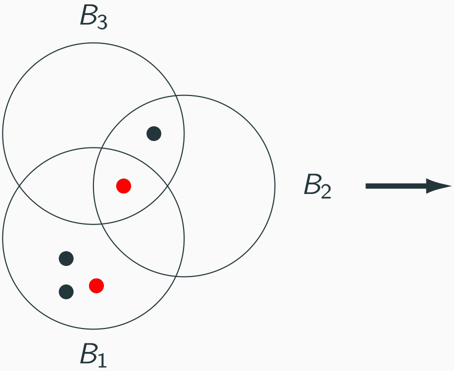

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

ZOE to p-Splitting Reduction

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- Our goal is to reduce the ZOE problem to a p-splitting by using a polynomial time algorithm.

ZOE to p-Splitting Reduction

- Our goal is to reduce the ZOE problem to a *p*-splitting by using a polynomial time algorithm.
- We will use the matrix version of the *p*-splitting problem in order to more easily visualize this reduction.

Reduction

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- Given a ZOE, we want to add columns to each row (elements to each set) such that we select exactly a single 1 from each row of the ZOE while making a split.

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- Rather than describing the construction generally, we will instead go through an example.

Example Reduction

Let $p = \frac{1}{3}$ and a ZOE system be

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Example Reduction

First, we will make the right side of each row equal s_i which represents the number of 1's in each row i.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ \hline \vdots & & & & \end{bmatrix} \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ \vdots \end{bmatrix}$$

Example Reduction

Now we add columns so that in each row we add a number of 1's equal to $s_i - 1$. We'd like to select all of the green 1's which forces us to select exactly one of the 1's from A.

$$\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right] \mathbf{v} = \left[\begin{array}{c} 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

Example Reduction

Now we add the correct amount of reds so that $\left\lfloor \frac{1}{3} t_i \right\rceil = s_i$ where t_i is the total amount of 1's per row i.

$$
\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & \color{green}{1} & \color{green}{0} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{0} \\
1 & 1 & 1 & 0 & \color{green}{1} & \color{green}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right] \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

Example Reduction

Note that if we select the green columns to be in the splitter, and not the red columns, then to satisfy each equation we need exactly one 1 from each row of the original A:

$$
\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & \color{green}{1} & \color{green}{0} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{0} \\
1 & 1 & 1 & 0 & \color{green}{1} & \color{green}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right] \mathbf{v} = \left[\begin{array}{c} 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]
$$

Example Reduction

Now our goal is to add sets such that we are forced to select exactly the green columns to be in the splitter. First, make every entry below the original A be zero.

$$
\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & &
\end{array}\right]
\quad \mathbf{v} = \left[\begin{array}{c} 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ \\ \\ \end{array}\right]
$$

Example Reduction

Now, make $\binom{4}{2}$ of the first green columns be 1's:

$$
\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right] \mathbf{v} = \left[\begin{array}{c} 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]
$$

Example Reduction

Now, in every row corresponding to a new green 1, make every subset of two 1's from the red columns.

$$
\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right] \mathbf{v} = \left[\begin{array}{c} 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]
$$

Example Reduction

Now do the same thing for the other green column:

$$
\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right] \mathbf{v} = \left[\begin{array}{c}
2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1
\end{array}\right]
$$

Example Reduction

Note that this construction forces you to choose all of the green columns, and therefore exactly one black 1 from each row. This corresponds directly to a solution to ZOE.

$$\left[\begin{array}{cccc|cc|cccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right] \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Fraction Splittable

Fraction of configurations that are splittable

- Define $f(n, k)$ to be the fraction of configurations with n sets and k elements that are splittable
- We looked at the limits of f for both large n and large k

Theorem

Fix k. Then $\lim_{n\to\infty} f(n, k) = 0$.

proof sketch.

Fixed $k \implies$ fixed $\mathcal{P}(k)$. If you keep adding sets to a fixed number of elements, eventually you expect to see that some of the elements land in an unsplittable configuration for 3 of the sets. □

Fraction of configurations that are splittable

Theorem

Let $\epsilon > 0$ *and let* $k > n^{1+\epsilon}2^n D$. *Then* $\lim_{k\to\infty} f(n,k) = 1$.

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- We always believed that if k grew faster than n then the following conjecture would tell us the fraction approached 1:

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Conjecture

A configuration with no nonempty sectors is splittable.

Fraction of configurations that are splittable

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Conjecture

A configuration with no nonempty sectors is splittable.

- Unfortunately this remains just a conjecture.

Something close

Proposition

Let D be Spencer's bound for the discrepancy of an n-set collection, i.e. $K\sqrt{n}$. If each of the 1-fold regions in the collection contains at least D elements, then the collection is splittable.

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Proposition

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- We will show that a configuration is splittable if k is big enough, since the 1-folds will likely have D elements.
- Specifically, we will show that $k > n^{1+\epsilon}2^n D$.

Proof

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- If the average number of elements in each monofold is many standard deviations above D, then the configuration is very likely splittable

Chebyshev's Inequality

- Chebyshev's Inequality states $Pr\left[|X-\mu| \geq t\sigma\right] \leq \frac{1}{t^2}$

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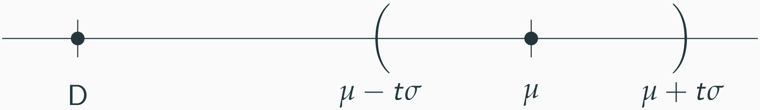

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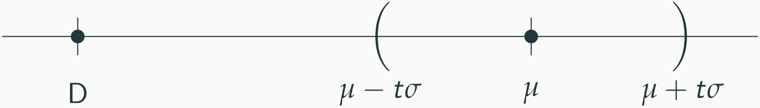

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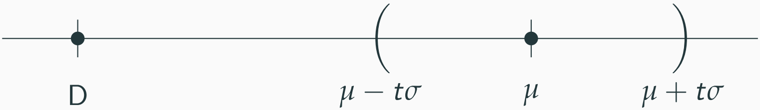

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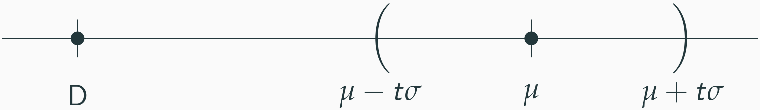

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- QED

References

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Thanks!

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