Generalizing the WHIRLPOOL Hash Function

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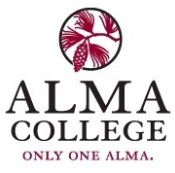
D. Nikolov, L. Wells

July 31, 2013



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- Given a message m_1, it should be hard to find a message m_2 that has the same hash as m_1.
- It should be hard to find *any* two messages m_1 and m_2 that have the same hash.

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Practical Hash Functions: MD4, MD5, SHA-1, SHA-2, WHIRLPOOL.

How Whirlpool Works - I

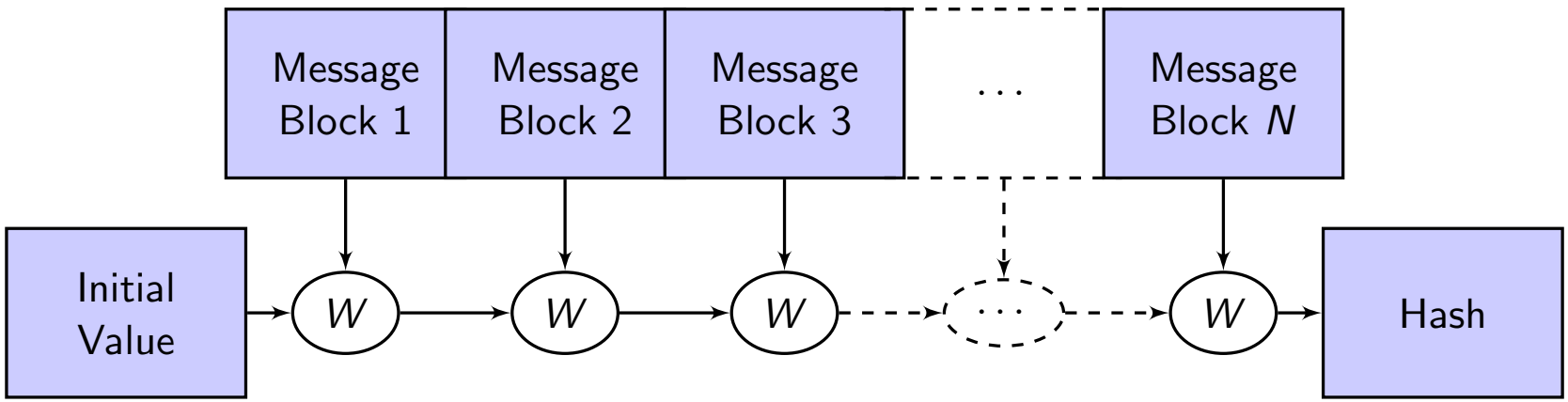

Figure : The Merkle-Damgård Construction

How Whirlpool Works - II

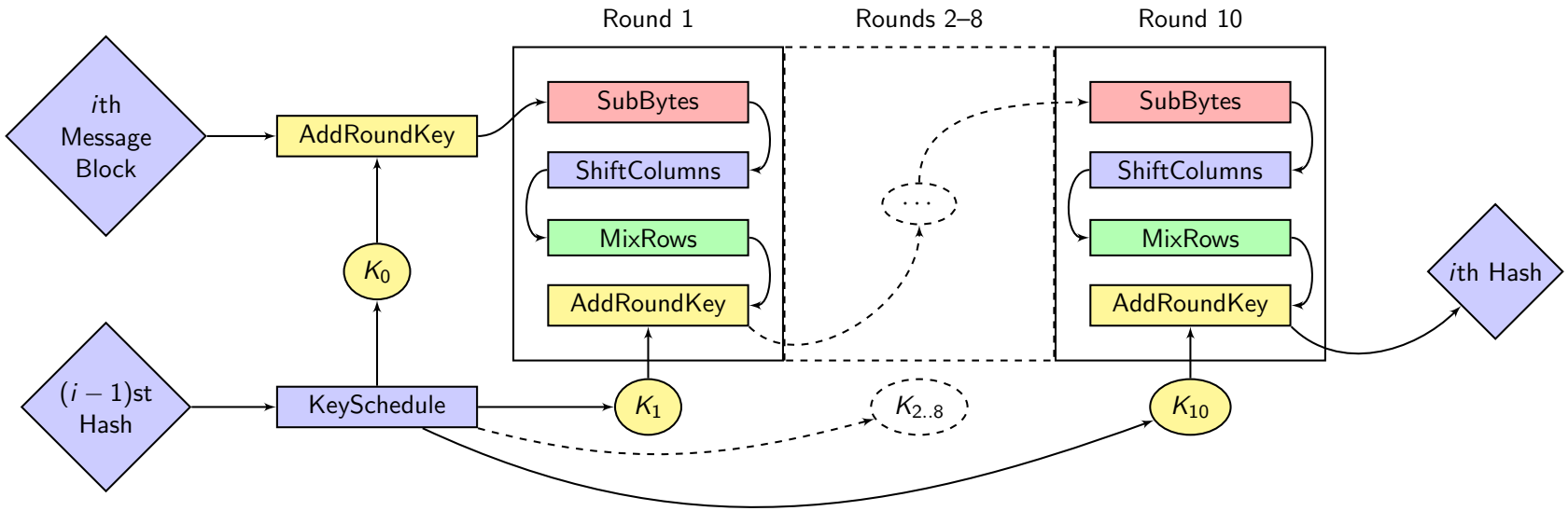

Figure : The Block Function W

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To evaluate the security of Whirlpool, we need to know whether it *forms* a group, and what the size of the group that it *generates* is.

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To find the parity of the complete Whirlpool function, we look at the parity of each sub-function.

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AddRoundKey ($\sigma[k]$) function

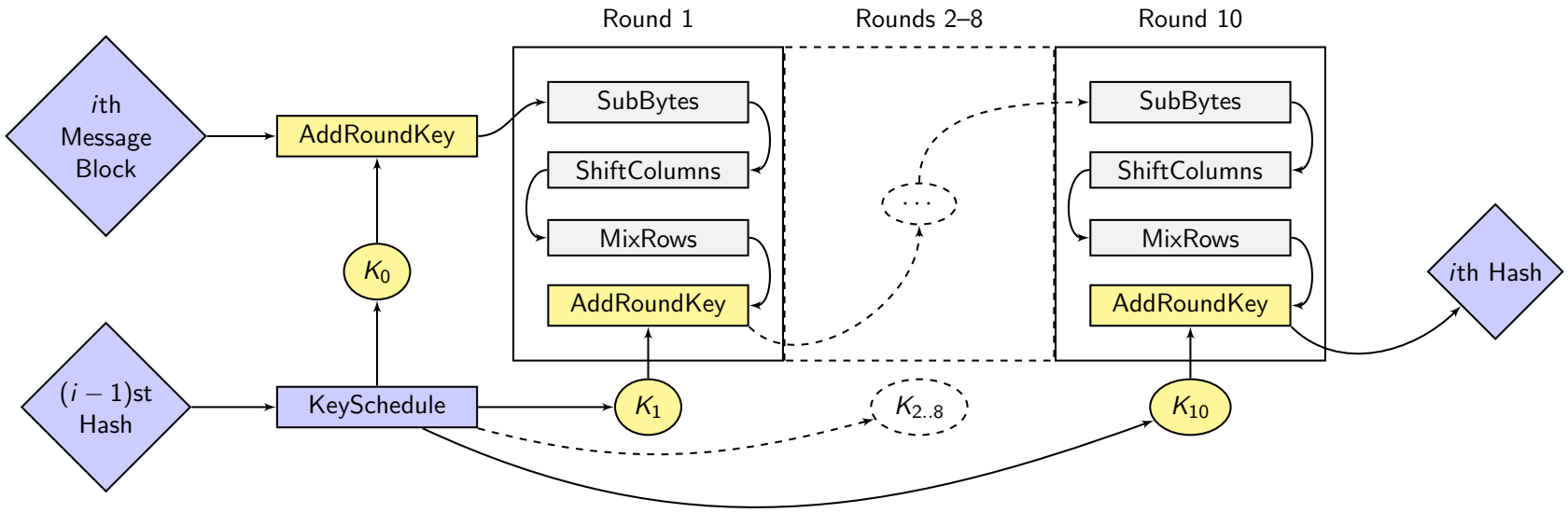

Figure : AddRoundKey

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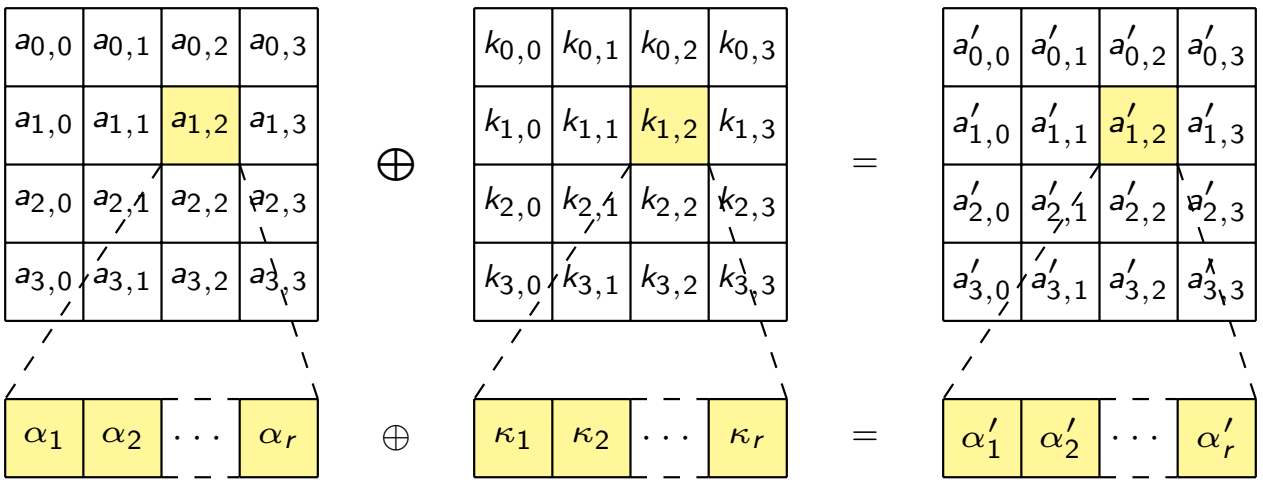

Figure : Example AddRoundKey Function

Parity of Add Round Key ($\sigma[k]$) function

Lemma

For all $k \in M_{m,n}(\mathrm{GF}(p^r))$ and $r, m, n > 1$, the function $\sigma[k]$ is an even permutation. [a]

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[a] L. Babinkostova, K. Bombardier, M. Cole, T. Morrell, and C. Scott, *Algebraic Structure of generalized Rijndael-like SP networks*, (2012) (arXiv: 1210.7942)

ShiftColumn (π) Function

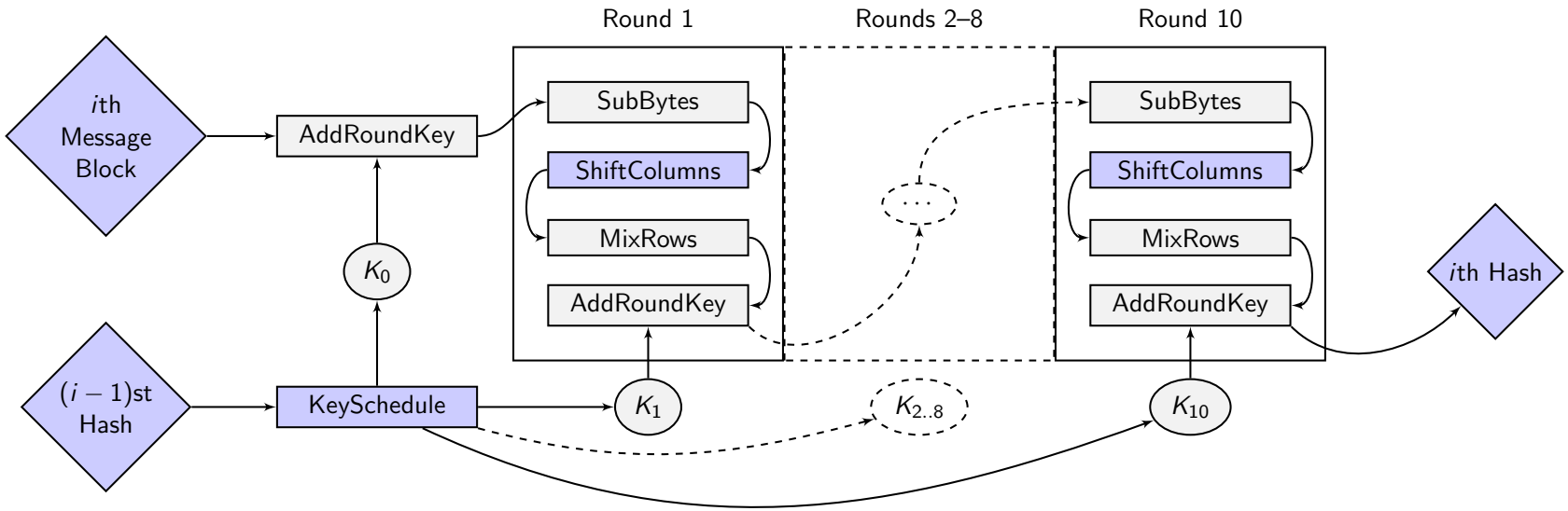

Figure : ShiftColumn

ShiftColumn (π) Function

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Definition

Let $\pi : M_{m,n}(\mathrm{GF}(p^r)) \rightarrow M_{m,n}(\mathrm{GF}(p^r))$ denote the mapping for which there is a function $s : \{0, \ldots, n-1\} \rightarrow \{0, \ldots, m-1\}$ such that $\pi(a) = a'$ if and only if $a'_{i,j} = a_{(i+s(j)) \bmod m,\ j}$ for all $0 \leq i < m$, $0 \leq j < n$. The mapping π is called a "Shift Column" function.

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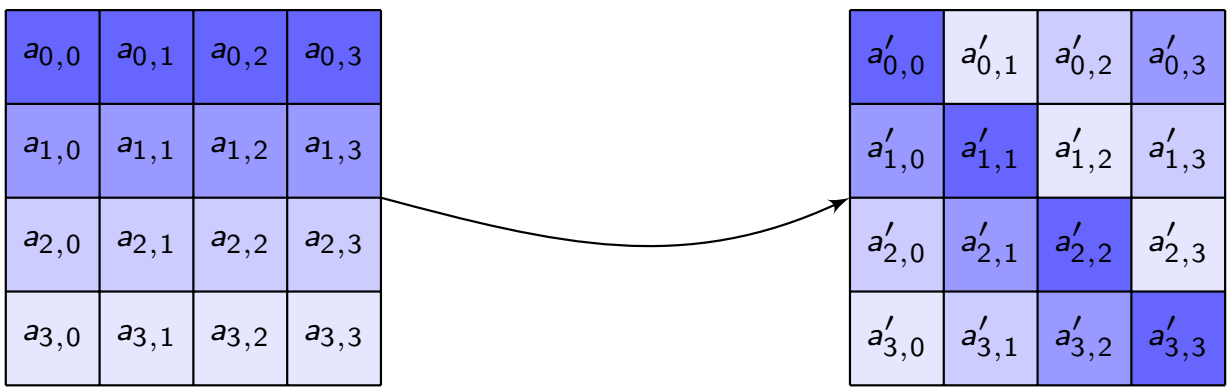

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- Generalizing for an arbitrary matrix with arbitrary shifts $\rightarrow$ look at each column separately

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$$n\left(2^k \prod_{l=1}^{N} p_\ell^{k_\ell}\right) = \sum_{\substack{i_0 \in \{0,\ldots,k-1\} \\ i_1 \in \{0,\ldots,k_1-1\} \\ \vdots \\ i_N \in \{0,\ldots,k_N-1\}}} (-1)^{\sum_{\ell=0}^{N} i_\ell} N\left(2^{k-i_0} \prod_{l=1}^{N} p_\ell^{k_\ell - i_\ell}\right)$$

Counting Cycles of the WHIRLPOOL "Shift Column"

The Möbius Inversion Theorem

If g and f are arithmetic functions, $g(c) = \sum_{d|c} f(d)$ for every integer $c > 1$, then $f(c) = \sum_{d|c} \mu(d) \cdot g\left(\frac{c}{d}\right)$ for every integer $c > 1$.

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- $\mu(d) = 1$ if d is a square-free positive integer with an even number of prime factors.
- $\mu(d) = -1$ if d is a square-free positive integer with an odd number of prime factors.
- $\mu(d) = 0$ if d has a squared prime factor.

How the Möbius function simplifies

Without Möbius Inversion Formula:

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Alternative ways of thinking about Shift Column

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Preliminary results on finite fields

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Theorem

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Theorem

If $p(x)$ is an irreducible polynomial over a finite field $\mathrm{GF}(p^m)$, then $p(x)$ has no multiple roots.[a]

[a] J. A. Gallian, *Contemporary Abstract Algebra*, **Huston Mifflan Company**, (1992).

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This implies that there is a mapping from the set of elements of $\mathrm{GF}(p^m)$ to the set of monic irreducible polynomials of degree m over $\mathrm{GF}(p)$ that are not in any field $\mathbb{F}$ with $\mathrm{GF}(p) \subseteq \mathbb{F} \subset \mathrm{GF}(p^m)$.

The connection with monic irreducible polynomials

Every subfield of $\mathrm{GF}(p^m)$ is of the form $\mathrm{GF}(p^c)$ where $c|m$. So, the number of elements in $\mathrm{GF}(p^m)$ that are not in any proper subfield of $\mathrm{GF}(p^m)$ is

$$\sum_{c|m} \mu(c) p^{\left(\frac{m}{c}\right)}$$

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Thus, a monic irreducible polynomial of degree c has factorization

$$(x - \alpha)(x - \alpha^p)(x - \alpha^{p^2}) \ldots (x - \alpha^{p^{c-1}}).$$

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There is a bijection between the set of all monic irreducible polynomials over $\mathrm{GF}(p^c)$ and the set of c – *element* sets

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Any two irreducible polynomials have distinct roots, their corresponding sets are disjoint.

Therefore, we have that the number of monic irreducible polynomials of degree c is

$$\frac{1}{c} \sum_{d|c} \mu(d) p^{\frac{c}{d}}.$$

The connection with monic irreducible polynomials

Theorem

For every prime power $p > 1$ and every positive integer c there exists a primitive normal basis of $\mathrm{GF}(p^c)$ over $\mathrm{GF}(p)$.[a]

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Thus, each element $x \in \mathrm{GF}(p^c)$ can be uniquely written as

$$x = a_0\alpha + a_1\alpha^p + \ldots a_{c-1}\alpha^{p^{c-1}}$$

where $a_0, a_1, \ldots a_{c-1} \in \mathrm{GF}(p)$.

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is a permutation on $GF(p^c)$, which fixes exactly the elements of $GF(p^d)$, where $d|c$.

The Frobenius Map, cont'd

The elements of $GF(p^d)$ thus satisfy:

$$f^d(x) = x$$

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$$x = \sum_{j=0}^{c-1} a_j \alpha^{p^j}$$

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and:

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$$f^d(x) = \sum_{j=0}^{p^c-1} a_j \alpha^{p^{j+d}}$$

so: $\forall j \in \{0, ..., c-1\}, a_j = a_{j+d}$

The Frobenius Map, cont'd

The elements of $GF(p^d)$ thus satisfy:

$$f^d(x) = x$$

and:

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Note that applying f^d to $x = (a_0, a_1, ..., a_{c-1})$ corresponds to applying a shift of d on the entries of x.

The Frobenius Map, cont'd

So the fixed points of f^d are in bijection with the lists $(a_0, a_1, ..., a_{c-1}) \in (GF(p))^c$ that are fixed after a shift of d on its entries.

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$\frac{C(f,c)}{c}$ is the number of c-cycles.

The Frobenius Map, cont'd

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For 2-cycles:

$$\begin{aligned} &\text{If } p = 1 \pmod 4, && \text{then the parity is even,} \\ &\text{If } p = 3 \pmod 4, && \text{if } gcd(k,m) \text{ is even, the parity is even,} \\ & && \text{if } gcd(k,m) \text{ is odd, the parity is odd.} \end{aligned}$$

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MixRow (ρ) Function

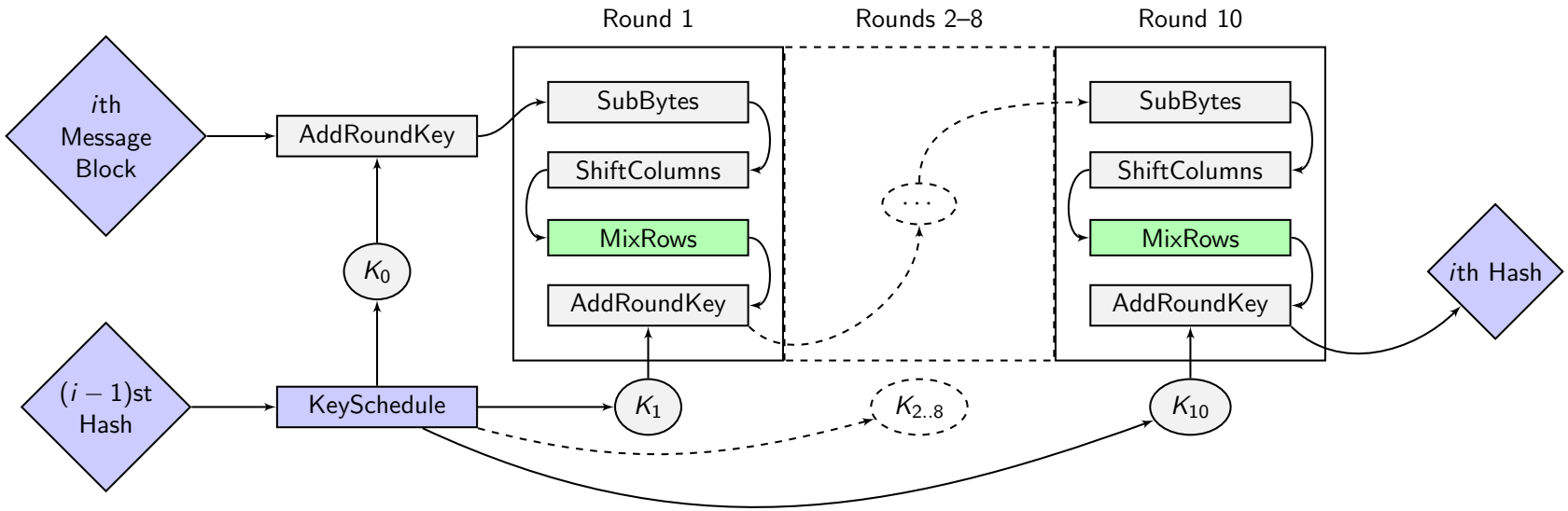

Figure : MixRow

MixRow (ρ) Function

Definition

Let $\rho : M_{m,n}(\mathrm{GF}(p^r)) \to M_{m,n}(\mathrm{GF}(p^r))$ is a mapping defined as the parallel application of n "row" mappings $\rho_i : M_{1,n}(\mathrm{GF}(p^r)) \to M_{1,n}(\mathrm{GF}(p^r))$ defined by $\rho(a) = a'$ if and only if $a'_i = \rho_i(a_i)$ for all $0 \leq i < m$, where each ρ_i is given by $\rho_i(x) = x \cdot C$ for all $x \in M_{1,n}(\mathrm{GF}(p^r))$, where $C \in M_{n,n}(\mathrm{GF}(p^r))$ is an invertible diffusion matrix.

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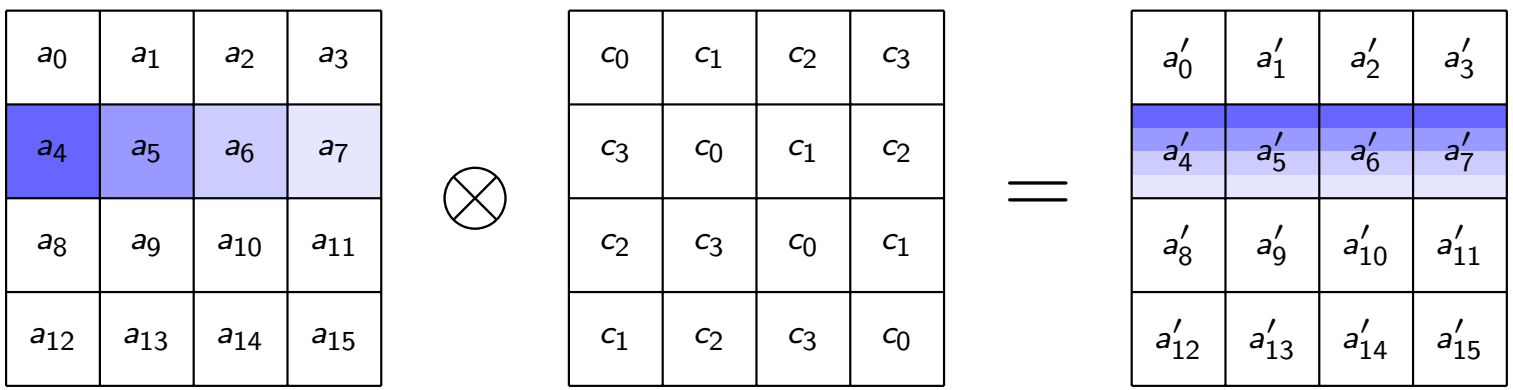

Figure : Example of MixRow Function

Previous Results

Theorem

Let C be an invertible MDS matrix

MixRow is an odd permutation if and only if p, m and $\frac{p^{rn}-1}{|\langle C \rangle|}$ are odd. [a]

[a] L. Babinkostova, K. Bombardier, M. Cole, T. Morrell, and C. Scott, *Algebraic Structure of generalized Rijndael-like SP networks*, (2012) (arXiv: 1210.7942)

Previous Results cont.

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ϕ maps a matrix A to its corresponding representative polynomial (the polynomial whose coefficients are the entries in the matrix). [1]

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The Isomorphism, cont'd

We factor the polynomial $(x^n - 1)$ as $x^n - 1 = f_1^{n_1} \cdot f_2^{n_2} \cdots f_k^{n_k}$, where $f_i (1 \leq i \leq k)$ are distinct monic irreducible polynomials of $F[x]$.

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Then:

$$\mathcal{M} \cong \frac{F[x]}{\langle (x^n - 1) \rangle} \cong \frac{F[x]}{\langle f_1^{n_1} \rangle} \times \frac{F[x]}{\langle f_2^{n_2} \rangle} \times \cdots \times \frac{F[x]}{\langle f_r^{n_k} \rangle}$$

Chinese Remainder Theorem

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$$\begin{aligned} f(x) &= a_1 \mod (f_1^{n_1}) \\ f(x) &= a_2 \mod (f_2^{n_2}) \\ &\vdots \\ f(x) &= a_k \mod (f_r^{n_k}) \end{aligned}$$

This $f(x)$ will be of the form

$$f(x) = c_{n-1}x^{n-1} + \cdots c_1 x^1 + c_0$$

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$$C = circ\{c_0, c_1, \cdots, c_{n-1}\}$$

Improved Result

Theorem

Let C be an invertible circulant matrix where $C \leftrightarrow (a_1, a_2, \cdots, a_k)$ and $a_i \in \frac{F[x]}{\langle f_i^{n_i} \rangle}$.

If $\sum_{i=1}^{n} c_i \not\equiv 1$, then MixRow is an odd permutation if and only if p, m and $\frac{p^{rn}-1}{\text{lcm}(\{\text{ord}(a_i)\}_1^k)}$ are odd.

If $\sum_{i=1}^{n} c_i \equiv 1$, then MixRow is an odd permutation if and only if p, m and $\frac{p^{rn}-p^r}{\text{lcm}(\{\text{ord}(a_i)\}_1^k)}$ are odd.

$|C|$ when n is prime

Theorem

Assume n is prime and $q = p^r$. Then the multiplicative group of $n \times n$ circulant matrices

$$C_n^*(GF(q)) \cong \mathbb{Z}_q^* \otimes \mathbb{Z}_{q^w}^{*\ \frac{n-1}{w}},$$

where $w = ord_n(q)$. [a]

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If q is a primitive root of n, then $\mathcal{C}_n^*(GF(q)) \cong \mathbb{Z}_q^* \otimes \mathbb{Z}_{q^{n-1}}^*$

When q is a primitive root of n

For $C \in \mathcal{C}_n^*(GF(q))$,

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Parity of $|C|$

Define a, b, c, d, e such that

$$\begin{aligned} q^{n-1} - 1 &= 2^a \cdot P_1 \\ q - 1 &= 2^b \cdot P_2 \\ k &= 2^c \cdot P_3 \\ \ell &= 2^d \cdot P_4 \\ |C| &= 2^e \cdot P_5 \end{aligned}$$

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Using the formula for $|C|$,

$$e = a+b-min(a,c)-min(b,d)-min((a-min(a,c)),(b-min(b,d)))$$

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$|C|$ is odd if and only if $c \geq a$ and $d \geq b$

Parity of MixRow

Theorem

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Min and Max of $|C|$

Let C be an invertible circulant MDS matrix.

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Generalizing for any w

For $C \in \mathcal{C}_n^*(GF(q))$,

$C \leftrightarrow (x, y_1, y_2, \cdots, y_{\frac{n-1}{w}}) = (g_1^{\ell}, g_2^{k_1}, g_2^{k_2}, \cdots, g_2^{k_{\frac{n-1}{w}}})$ for $x \in \mathbb{Z}_q^*$ and $y_i \in \mathbb{Z}_{q^w}^*$ and where w is the order of q mod n

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$$q^w - 1 = 2^a \cdot P_1$$
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Proposition

$|C|$ is odd if and only if $c_i \geq a$ and $d \geq b$

Generalizing cont.

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$|C|$ attains its maximum of $q^w - 1$ if and only if $\exists i \in [1, \frac{n-1}{w}]$ such that $\gcd(k_i, q^w - 1) = a, a|q - 1$ and $\gcd(a, \gcd(\ell, q - 1)) = 1$

Data Collected

q	$q-1$	n	$\max\{\lvert C\rvert\}$	$\lvert\mathcal{C}_n^*(\mathrm{GF}(q))\rvert$
2^4	15	2	30	240
		3	15	3375
		4	60	61440
		5	15	759375
		7	?	251535375
3^2	8	2	8	64
		3	24	648
		4	8	4096
		5	40	51200
		7	?	4239872

SubBytes (λ) Function

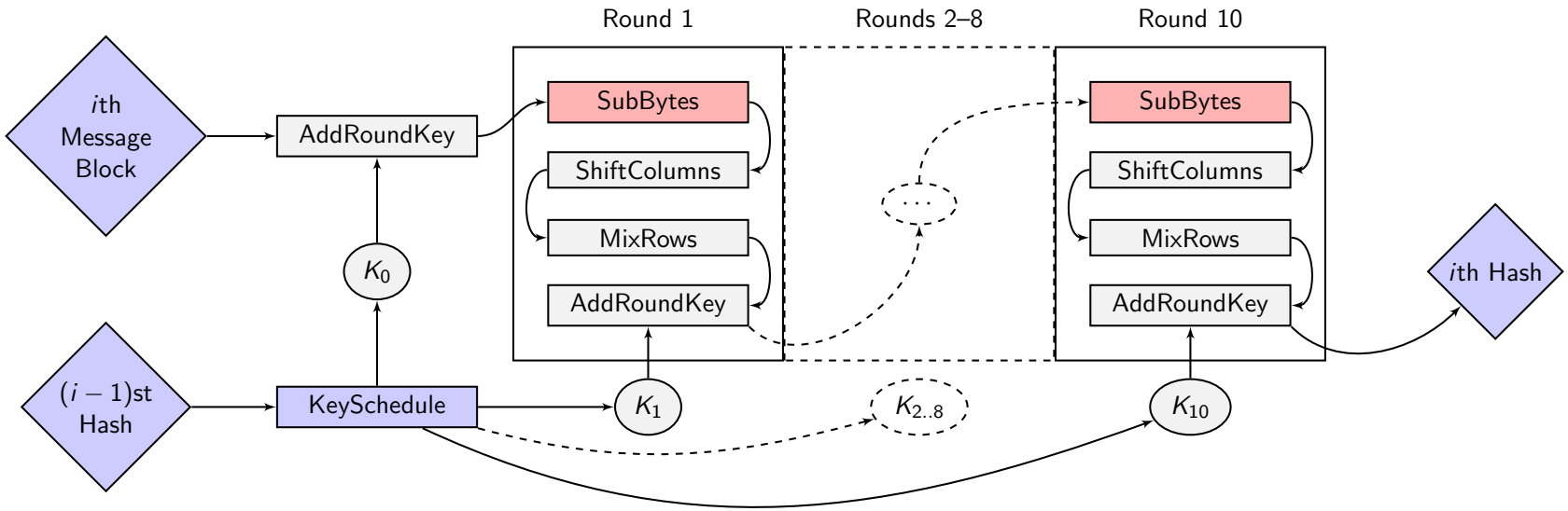
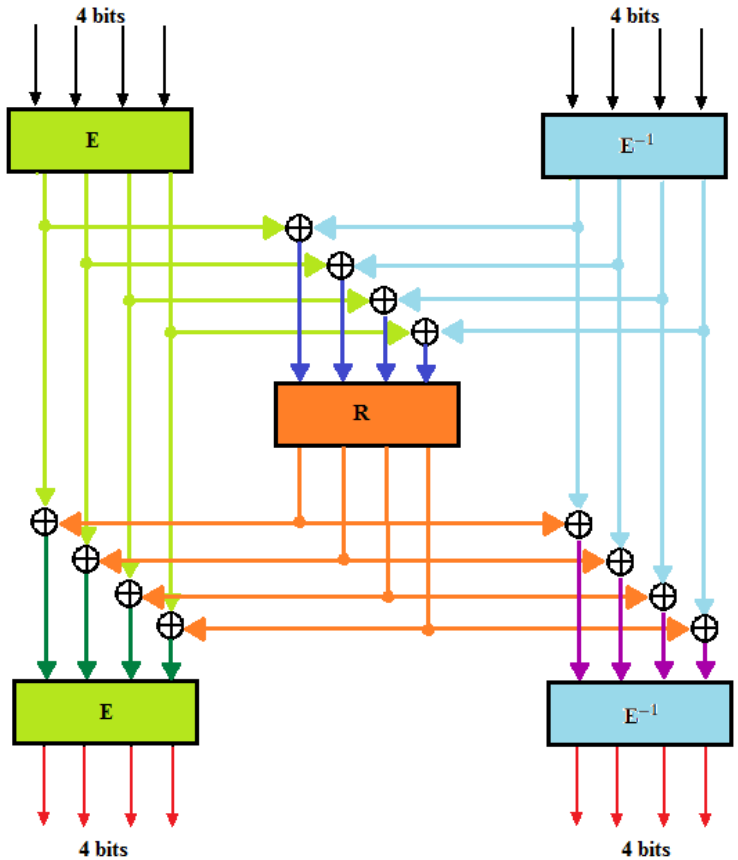


Figure : SubBytes

The Whirlpool S-Box



$E =$

u	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
$E(u)$	1	B	9	C	D	6	F	3	E	8	7	4	A	2	5	0

$R =$

u	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
$R(u)$	7	C	B	D	E	4	9	F	6	3	8	A	2	5	1	0

The E Mini-Box

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The underlying mapping of the WHIRLPOOL mini E-box is exponentiation in a finite field.

Let g be a generator of the multiplicative group $GF(p^r) \setminus \{0\}$ and $q(x)$ be a primitive polynomial in $\mathrm{GF}(p^r)$.

We consider the map $E : \mathrm{GF}(p^r) \longrightarrow \mathrm{GF}(p^r)$ with respect to the polynomial $q(x)$ and map $u \in GF(p^r) \setminus \{0\}$ to g^u, where $u \in \mathrm{GF}(p^r)$ is taken to be its numerical value

$$u = \sum_{i=0}^{r-1} u_i . p^i .$$

The E Mini-Box

Definition

$$E(u) = \begin{cases} 0 & \text{if } u = p^r - 1 \\ g^u & \text{otherwise} \end{cases},$$

where $u \in GF(p^r)$ and g is a generator of $\mathrm{GF}(p^r)^*$.

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The E Mini-Box

Definition

We define the permutation induced by the generator g as the mapping $u \longrightarrow g^u$, where u runs through all the elements $\{1, 2, \ldots, p^r - 1\}$ of the finite field.

$$\sigma_g = \begin{pmatrix} 0 & 1 & 2 & \cdots & p^r - 2 & p^r - 1 \\ 1 & g^1 & g^2 & \cdots & g^{p^r - 2} & 0 \end{pmatrix}$$

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Otherwise, we call g a "bad" generator.

Facts about Generators

Theorem

Let g be a generator $(\mathrm{mod}\ p)$, *σ_g be its induced permutation, and* $T = \{\tau_j = \{j, (p-1)-j\} | 1 \leq j < p-1, j \neq \frac{p-1}{2}\}$.

- $(\sigma_{g^{-1}}$ *has* $1-$ *cycle* $(a))$ *iff* $(\sigma_g((p-1)-a) = a)$
- $(\sigma_{g^{-1}}$ *has* $2-$ *cycle* $(a, b))$ *iff* $((a, (p-1)-a) = \tau_i, (b, (p-1)-b)\tau_j$, $i \neq j$, *and* $\sigma_g((p-1)-a) = b$, *and* $\sigma_g((p-1)-b) = a)$

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i.e., $\sigma_{p-g}(\frac{p+1}{2}) = g$.

Facts about Generators

Proposition

Let $v \in \mathbb{Z}_p^*$ *and* $d = \frac{p-1}{ord_p(v)}$, *then*

$\{u \in \mathbb{Z}_p \mid g^u = v \text{ for some } g\} = \{u'd \mid gcd(u', \frac{p-1}{d}) = 1\}$.

Furthermore, every u *is an pre-image of* v *exactly* $\frac{\phi(p-1)}{\phi(\frac{p-1}{d})}$ *times.*

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Furthermore, every u *is an pre-image of* v *exactly* $\frac{\phi(p-1)}{\phi(\frac{p-1}{d})}$ *times.*

Corollary

v *is its own pre-image iff* $gcd(v, p-1) = d$.

Facts about Generators

So given a prime p, there will be a fixed point for some generator if and only if there exists an element $v \in \mathbb{Z}_p^*$ where
$d = \frac{p-1}{ord_p(v)}$
and
$gcd(v, p-1) = d$.

Corollary

If v is a generator and is relatively prime to $p-1$, then v will be a fixed point for some generator.

Facts about Generators

Proposition

Let p be a prime, g be a generator dividing $p-1$, let $k_0 \in Z_p$ such that $gcd(g^{k_0}, p-1) = g^{k_0-1}$, then $g^{g^{k_0-1}}$ is a fixed point for some generator.

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Note that this is assuming the existence of a generator dividing $p-1$ and that $gcd(g, \frac{p-1}{g^{k_0-1}}) = 1$.

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So our question of the existence of fixed points is reduced to the existence of a prime generator for every finite cyclic group $\mathbb{Z}_p^*$!

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31 with primitive root
3 : (0 1 3 27 23 11 13 24 2 9 29 21 15 30)(4 19 12 8 20 5 26 18)
(6 16 28 7 17 22 14 10 25)
11 : (0 1 11 24 8 19 22 18 2 28 10 5 6 4 9 23 12 16 20 25 26 7 13 21
27 15 30)(3 29 17)(14)
12 : (0 1 12 4 28 14 18 8 9 15 30)(2 20 5 26 10 25 6)
(3 23 22 7 24 16 19)(11 21 29 13 17)(27)
13 : (0 1 13 11 3 27 23 24 2 14 19 21 15 30)(4 10 5 6 16 18)
(7 22 9 29 12 8)(17)(20 25 26 28)

Example for $p = 31$

17 : (0 1 17 21 23 13 3 15 30)
(2 10 25 6 8 18 16 14 20 5 26 9 27 29 11 22 19 24 4 7 12)(28)
21 : (0 1 21 29 3 23 17 24 16 10 5 6 2 7 11 12 4 18 8 14 28 9 15 30)
(13 22 20 25 26 19)(27)
22 : (0 1 22 10 5 6 8 28 18 16 9 27 29 24 4 20 25 26 14 7 21 23 3
15 30)(2 19 11 17 12)(13)
24 : (0 1 24 8 10 25 6 4 14 9 23 21 27 15 30)(2 18)
(3 29 22 28 19 17 13 12 16 7)(5 26 20)(11)

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- **ShiftColumn:** For 2-cycles:

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  If $p = 3 \pmod 4$, if $gcd(k, m)$ is even, the parity is even,

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- **MixRow:** Let $\mathcal{C} = \sum_{i=1}^{n} c_i$.
  - Odd permutation if and only if p, m and $\dfrac{p^{rn} - 1}{\mathrm{lcm}(\{\mathrm{ord}(a_i)\}_1^r)}$ are odd when $\mathcal{C} \not\equiv 1$.
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- **SubBytes:** The standard Whirlpool SubBytes in $\mathrm{GF}(2^8)$ is even.

Future Work

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- Find the group generated by the generalized WHIRLPOOL function.

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- Boise State Mathematics Department

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Questions?