Uniform Splitting Families

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Presentation Outline

The Discrete Log Problem

Problem: Let p be a prime. Let a be a primitive root of the multiplicative group $\mathbb{Z}_p^*$ and b an element of $\mathbb{Z}_p^*$. Then the **discrete log problem** is the problem of finding the exponent x so that $a^x = b$.

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In general this is thought to be a hard problem but in some special circumstances clever algorithms can make it much easier.

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As it turns out the answer is yes, and the mathematical objects we have been studying provide a (relatively) efficient solution in $O(m\binom{m/2}{t/2})$ where m is the number of digits in the binary representation of p.

Presentation Outline

Formal Definition

Definition

Let m and t be fixed integers with $0 < t \leq m$. Let X be a set of size m. Let a subset of X be a **block**. A family $\mathcal{F}$ is a set of blocks, and is called an $(\mathbf{m}, \mathbf{t})$-**splitting family** if for all sets $Y \subseteq X$ such that $|Y| = t$, there exists a block $b \in \mathcal{F}$ so that $|b \cap Y| = \lfloor t/2 \rfloor$. We call $\mathcal{F}$ **uniform** if all blocks have size $\lfloor m/2 \rfloor$.

Examples

- An (8,2)-uniform splitting family

$$\mathcal{F} = \{\{1,2,3,4\},\{1,2,5,6\},\{1,3,5,7\}\}$$

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X	1	2	3	4	5	6	7	8
b_1	1	2	3	4				
b_2	1	2			5	6		
b_3	1		3		5		7	

Examples

- An (8,4)-uniform splitting family

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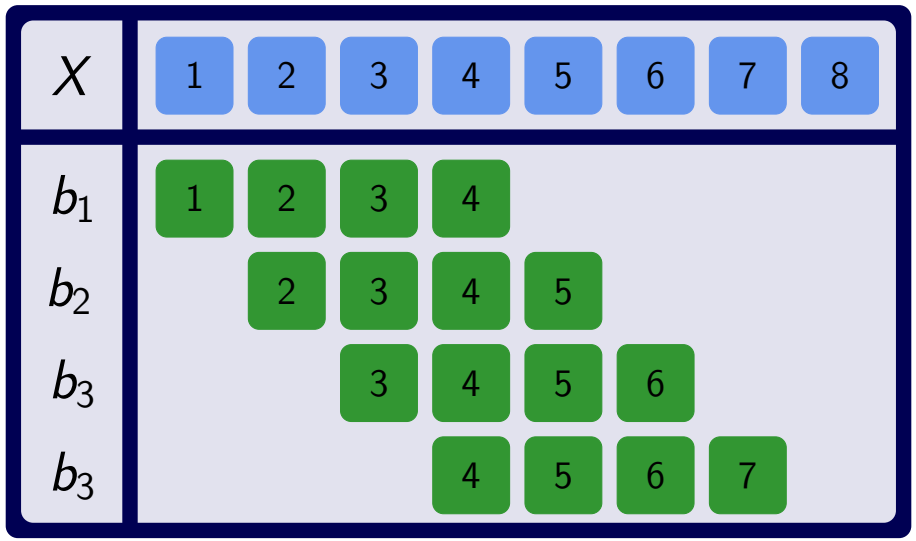

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$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

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Note, in particular, that all minimum-sized splitting families are also minimal.

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$(m, 2)$-uniform splitting families

We spent the first half of the summer focusing largely on $(m, 2)$ uniform splitting families, concluding with the following theorem

Theorem

Let $\mathcal{F}$ be a minimal $(m, 2)$ uniform splitting family with $m \geq 6$. Then

$$\lceil \log_2(m) \rceil \leq |\mathcal{F}| \leq m - 3.$$

Uniform splitting families for larger t

Results for $t > 2$ are remarkably harder to come by, but after some work we were able to prove the following upper bound on minimum-sized families using methods presented in [2].

Theorem

Let m and t be even. Let $\mathcal{F}$ be a minimum-sized (m, t) uniform splitting family. Then

$$|\mathcal{F}| \leq \frac{t \log_2(m)}{-\log_2\left(1 - \binom{t}{t/2}\left(\frac{1}{2}\right)^t\right)}.$$

An analogous result holds for m and/or t odd.

A comparison

Below is a comparison of this log bound with the currently best known bounds on maximum size.

m	t	Log bound	Best known [2]
8	4	18	3
10	4	20	5
12	4	22	6
14	4	23	7
16	4	24	6
18	4	25	9
20	4	26	8
22	4	27	11
24	4	28	9
26	4	28	13
38	4	29	14
30	4	29	12

Uniform splitting families for larger t

Bounds on minimal splitting families turned out to be even trickier but we were ultimately able to prove the following upper bound.

Theorem

Let m and t be even. Let $\mathcal{F}$ be a minimal (m, t) uniform splitting family. Then

$$|\mathcal{F}| \leq \frac{2}{3}\binom{m}{t}.$$

Witnesses

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Definition

Let $\mathcal{F}$ be a splitting family, and let Y be a t-set that is split by exactly one block $b \in \mathcal{F}$. Then, we say that Y is a witness for b, and for our splitting family $\mathcal{F}$.

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What do these look like?

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$$\left[\begin{array}{cc|cccc|cc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Is this the best?

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This is the largest construction we've come up with for any t, indicating that it is probably the upper rather than lower bounds that needs improvement.

References

[1] D. R. Stinson. Some baby-step giant-step algorithms for the low Hamming weight discrete logarithm problem. *Math. Comp.*, 71(237):379-391 (electronic), 2002.

[2] A. C.H. Ling, P. C. Li, G. H. J. van Rees. Splitting systems and separating systems. *Discrete Math* 279 (2004) 355-368,

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Questions?