Quantifying CDS Sortability of Permutations Using Strategic Piles

Context Directed Swag
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Motivation

Ciliates

- Sorting is a fundamental to many natural, industrial, and commercial processes.

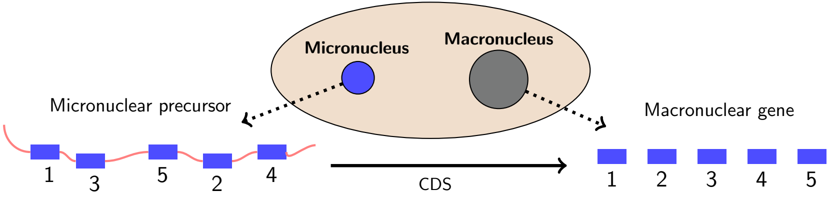

Figure: CDS as used in ciliates

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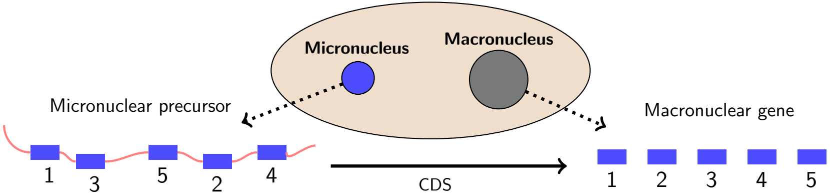

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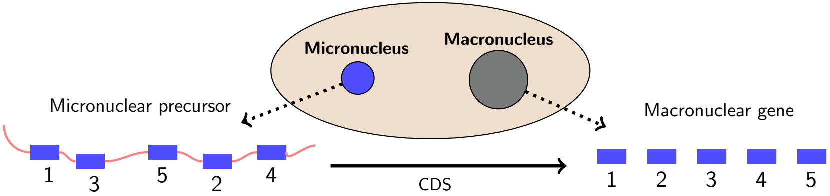

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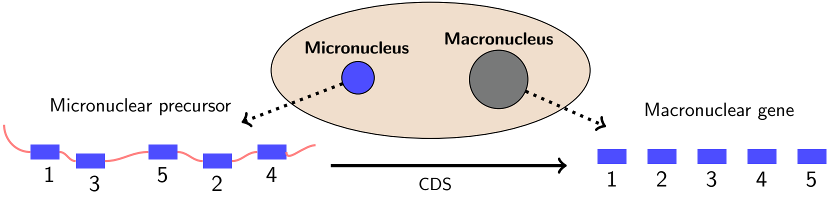
- A natural example: the one-celled organisms known as a ciliates.
- A ciliate must sort DNA in its micronuclei in order to create a macronucleus.
- Biologists hypothesize that ciliates use iterations of the context directed swap (CDS) sorting operation to rearrange their DNA [3].



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Preliminaries

Context Directed Swaps

Consider $\pi \in S_n$ where $\pi = [a_1\ a_2\ \ldots\ a_n]$. To each a_i, assign a left pointer, $\langle a_i - 1, a_i \rangle$, and a right pointer, $\langle a_i, a_i + 1 \rangle$. The CDS operation uses two pointers p and q in the order $p \ldots q \ldots p \ldots q$ and exchanges the elements flanked by these pointers.

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There are no remaining pointers in the order $p \dots q \dots p \dots q$.

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  - Then $C_\pi = Y_\pi \circ X$.
- If C_π contains a cycle of the form $= (0\ \ldots\ n\ b_1\ \ldots\ b_k)$, then $SP(\pi) = \{b_1, \ldots, b_k\}$. Otherwise, π is sortable and $SP(\pi) = \emptyset$.

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- Often, our results depend on the order of the strategic pile elements, so assume $\{b_1, \ldots, b_k\}$ occur chronologically in C_π.

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Since 0 and 5 are not in the same cycle, the permutation has empty strategic pile and so is sortable.

Research Goal

Our goal is to count the number of permutations in S_n that, when operated on by CDS, can result in k different fixed points. In a sense, we count the number of permutations with a specific size strategic pile in order to quantify permutations by how "close" they are to being sortable.

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If $\pi \in S_n$ has a full strategic pile, then $C_\pi = (0\ n\ b_1\ \ldots\ b_{n-1})$.

Thus, we can count all C_π of this form.

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Main Result

Theorem

For an even n, the number of permutations in S_n with full strategic pile is $\frac{2(n-1)!}{n}$.

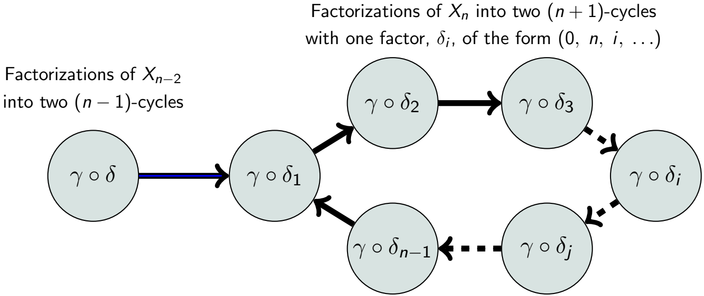

Figure: A graphical explanation of the different transformations that will yield the formula above.

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Now we will start with two factors of X_{n-2} that are $(n-1)$-cycles and injectively map to all factorizations of X_n two $(n+1)$-cycles where the right-most factor is of the form $(0\ n\ \ldots)$.

First Transformation

Let $\lambda_n = (0\ n\ 1)$ and $c_n = (1\ 2\ \ldots\ n-1)$.

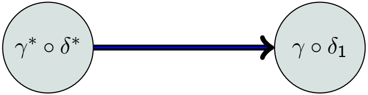
Lemma

Let $X_{n-2} = \gamma \circ \delta$ where γ and δ are $(n-1)$-cycles. We define the following:

$$\gamma_1 = \lambda_n \circ c_n \circ \gamma \circ (c_n)^{-1}$$
$$\delta_1 = c_n \circ \delta \circ (c_n)^{-1} \circ \lambda_n$$

Then γ_1 and δ_1 are $(n+1)$-cycles, δ_1 is of the form $(0\ n\ 1\ \ldots)$, and $X_n = \gamma_1 \circ \delta_1$.

First Transformation Explanation



We have from [3] that there are $\frac{2(n-2)!}{n}$ factorizations on the left, and since the map defined is injective, there are this many on the right as well.

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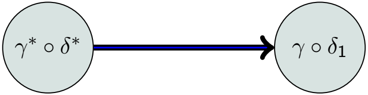

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Note that we have only mapped to δ factors of the from $(0\ n\ 1\ \ldots)$, and now must retrieve other factors of the form $(0\ n\ i\ \ldots)$

Second Transformation

We now show that another injective map will allow for the inclusion of δ factors that are of the form $(0\ n\ i\ \ldots)$ for all $2 \leq i \leq n-1$.

Lemma

Assume γ_i and δ_i are $(n+1)$-cycles, δ_i is of the form $(0\ n\ i\ \ldots)$, and $\gamma_i \circ \delta_i = X_n$. Define:

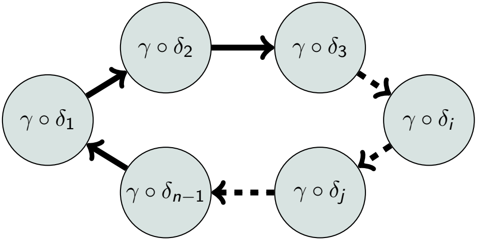
$$\gamma_{i+1} = r_n \circ c_n \circ \gamma_i \circ (c_n)^{-1}$$
$$\delta_{i+1} = c_n \circ \delta_i \circ (c_n)^{-1}$$

where $r_n = (2\ 1\ n)$. Then, γ_{i+1} and δ_{i+1} are $(n+1)$-cycles, δ_{i+1} is of the form $(0\ n\ i+1\ \ldots)$, and $\gamma_{i+1} \circ \delta_{i+1} = X_n$.

Lemma

For any $1 < i < n$, $\delta_i \neq \delta_1$, and $\delta_n = \delta_1$.

Second Transformation Explanation



Since all maps are injective, there are still $\frac{2(n-2)!}{n}$ factorizations in each circle. Also, all of these factors are distinct.

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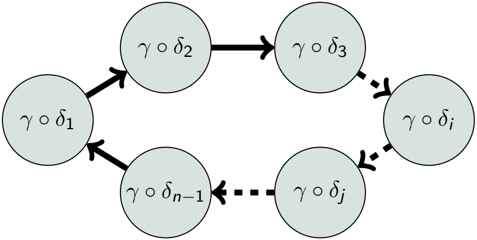

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The third Lemma stated shows that after this map is applied the $(n-1)^{st}$ time, the result will return to circle holding the δ_1 factorizations.

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- This is sufficient to show that all the factorizations have been counted.
- Thus there are a total of $n-1$ injective maps, all of whom have a size of $\frac{2(n-2)!}{n}$ [3]. So we multiply by $n-1$ to get the final count of $\frac{2(n-1)!}{n}$.

Strategic Piles of Size k

Form for Strategic Piles of Size k

Permutations with strategic pile $\{b_1, \ldots, b_k\}$ take the form:

$$[b_k + 1 \ \ldots \ b_{x_1} \ \underbrace{b_{x_1 - 1} + 1 \ \ldots \ b_{x_2}}_{\text{merge if equal}} \ b_{x_2 - 1} + 1 \ \cdots \ \underbrace{b_{x_{k-1}} \ b_{x_{k-1} - 1} + 1}_{\text{pair}} \ \ldots \ b_1].$$

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Together, the entries b_{x_i} $b_{x_i - 1} + 1$ are called a **pair**. A **merge** occurs when $b_{x_i - 1} + 1 = b_{x_{i+1}}$.

Example

Consider the permutation $\pi = [5\ 2\ 4\ 6\ 3\ 7\ 1]$ with $SP(\pi) = \{1, 5, 4\}$.

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$$\begin{array}{lccccccccl} [& 5 & 2 & 4 & 6 & 3 & 7 & 1 &] \\ [& b_3 + 1 & b_1 + 1 & b_3 & b_2 + 1 & 3 & 7 & b_1 &] \\ & b_2 & & & & & & & \end{array}$$

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Because $b_3 + 1 = b_2$, there is one merge in this permutation.

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5	$(n-5)!\,[\binom{n-6}{4}4! + \binom{n-6}{3}\_\_\_ + \binom{n-6}{2}\_\_\_ + \binom{n-6}{1}\_\_\_ + \binom{n-6}{0}\_\_\_\ ]$

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4	$(n-4)!\,[\binom{n-5}{3}3! + \binom{n-5}{2}16 + \binom{n-5}{1}16\ ]$
5	$(n-5)!\,[\binom{n-6}{4}4! + \binom{n-6}{3}90 + \binom{n-6}{2}130 + \binom{n-6}{1}80 + \binom{n-6}{0}90\ ]$

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1	$(n-1)!\,[\binom{n-2}{0}0!\ ]$
2	$(n-2)!\,[\binom{n-3}{1}1!\ ]$
3	$(n-3)!\,[\binom{n-4}{2}2! + \binom{n-4}{1}3 + \binom{n-4}{0}3\ ]$
4	$(n-4)!\,[\binom{n-5}{3}3! + \binom{n-5}{2}16 + \binom{n-5}{1}16\ ]$
5	$(n-5)!\,[\binom{n-6}{4}4! + \binom{n-6}{3}90 + \binom{n-6}{2}130 + \binom{n-6}{1}80 + \binom{n-6}{0}90\ ]$
6	$(n-6)!\,[\binom{n-7}{5}5! + \binom{n-7}{4}\underline{\quad\quad} + \binom{n-7}{3}\underline{\quad\quad} + \binom{n-7}{2}\underline{\quad\quad} + \binom{n-7}{1}\underline{\quad\quad}\ ]$

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6	$(n-6)!\,[\binom{n-7}{5}5!+\binom{n-7}{4}576+\binom{n-7}{3}1116+\binom{n-7}{2}1080+\binom{n-7}{1}540\ ]$

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where $c_{k,l}$ is called a **merge number** and represents the number of ways $k-1$ pairs can be ordered and have l merges

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k	$(n-k)!\,[\binom{n-(k+1)}{k-1}(k-1)! + \binom{n-(k+1)}{k-2}c_{k,1} + \cdots + \binom{n-(k+1)}{1}c_{k,k-2}\ ]$

where $c_{k,l}$ is called a **merge number** and represents the number of ways $k-1$ pairs can be ordered and have l merges

Strategic Piles of Size k

Theorem

The number of permutations of length n with strategic pile size k is

$$(n-k)!\sum_{i=0}^{k-1} c_{k,i}\binom{n-(k+1)}{k-(i+1)}$$

for k odd, and

$$(n-k)!\sum_{i=0}^{k-2} c_{k,i}\binom{n-(k+1)}{k-(i+1)}$$

for k even.

Strategic Piles of Size k

Formula: k odd

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  - The binomial coefficients count the ways to place these ordered pairs.

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2. $[b_3 + 1, \ldots b_3, b_2 + 1, \ldots b_2, b_1 + 1, \ldots b_1]$

Thus we can count the number of places a merge can occur in each form. A merge can occur between any of the pairs in the first form, and between none and the second. Therefore $c_{3,1} = 3 + 0 = 3$.

Merge Graphs

Given a permutation, we can define a directed graph to represent the merges present in that permutation. Let each strategic pile element be a vertex. There is an edge from b_i to b_j if and only if $b_i + 1 = b_j$

Merge Graphs

For example, consider the following permutation with strategic pile $\{9, 7, 4, 3, 2\}$:

$$
\begin{array}{cccccccccccc}
[& 3 & 5 & 1 & 7 & 10 & 2 & 4 & 8 & 6 & 9 &] \\
 & b_4 & & & b_2 & & b_5 & b_3 & & & b_1 & \\
 & b_5+1 & b_3+1 & & & b_1+1 & & b_4+1 & b_2+1 & & &
\end{array}
$$

Merge Graphs

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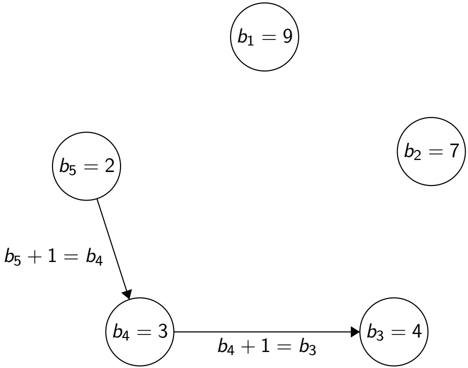
[	3	5	1	7	10	2	4	8	6	9	]
	b_4			b_2		b_5	b_3			b_1	
	$b_5 + 1$	$b_3 + 1$			$b_1 + 1$		$b_4 + 1$	$b_2 + 1$			

The corresponding merge graph would be:



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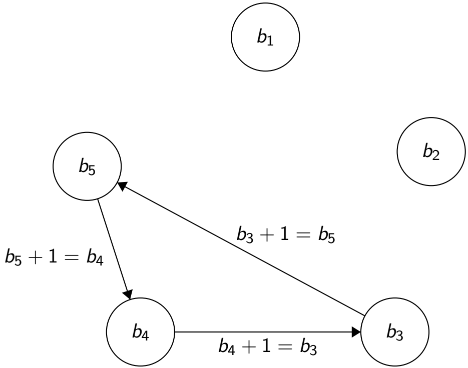

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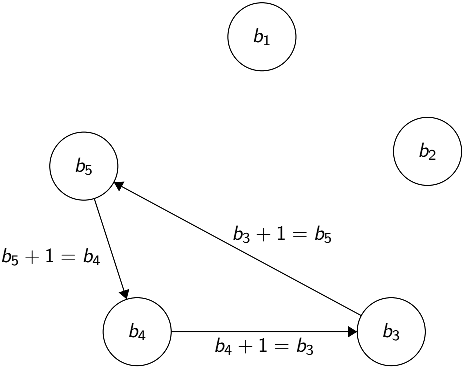

we know it cannot represent a merge graph. If all three of the equalities given are true, $b_5 + 1 = b_5 - 2$, a contradiction

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- For example, if we are considering permutations with strategic pile of size 4, and we know that the strategic pile elements appear in the order

  $$[b_5{+}1, \ldots, b_5, b_4{+}1, \ldots, b_3, b_2{+}1, \ldots, b_2, b_1{+}1, \ldots, b_4, b_3{+}1, \ldots, b_1]$$

  we know that it's possible $b_4 + 1 = b_3$, but that $b_4 + 1$ cannot equal b_2.

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  we know that it's possible $b_4 + 1 = b_3$, but that $b_4 + 1$ cannot equal b_2.
- We represent these possible merges in a graph with a vertex for each strategic pile element. We draw an edge from b_i to b_j if and only if, given the pair ordering, $b_i + 1 = b_j$ is a possible merge. This is equivalent to saying that $b_i + 1$ appears just before b_j in the pair ordering.

Graph of Possible Merges - Example

$$[b_5+1,\ldots,b_5,b_4+1,\ldots,b_3,b_2+1,\ldots,b_2,b_1+1,\ldots,b_4,b_3+1,\ldots,b_1]$$

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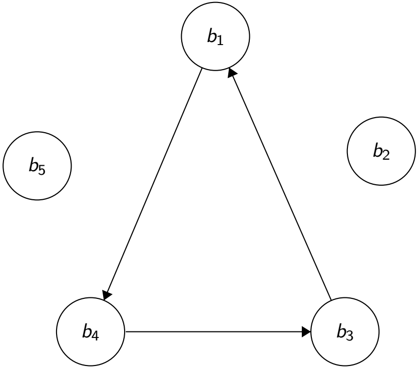

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- The map from a pair ordering to its graph of possible merges is injective.
- The merge graph for a permutation is an edge subgraph of the graph of possible merges for its pair ordering.

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2. For each of these partitions:
    1. Determine how many free pair orderings for strategic pile size k have the given cycle structure
    2. Use the inclusion-exclusion formula to figure out how many ways l merges can be picked with the given cycle structure

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    4. For general cases, this process can be carried out using Inclusion-Exclusion.
    5. Therefore, there are a total of $12[\binom{6}{3} - 4]$ free pair orderings with the given cycle structure

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3. We proceed similarly for each of the other listed cycle structures

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4. We sum the results for each cycle structure to get $c_{6,3}$:

$$c_{6,3} = \underbrace{24\left[\binom{6}{3} - 4\right]}_{[4,2]} + \underbrace{12\left[\binom{6}{3} - 2\right]}_{[3,3]} + \underbrace{48\left[\binom{5}{3}\right]}_{[5]} + \underbrace{15\left[0\right]}_{[2,2]} + \underbrace{20\left[0\right]}_{[3,3]} = 1080$$

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- Assuming this is true, this step of the algorithm is no faster than $\Omega(e^{\sqrt{k}})$.

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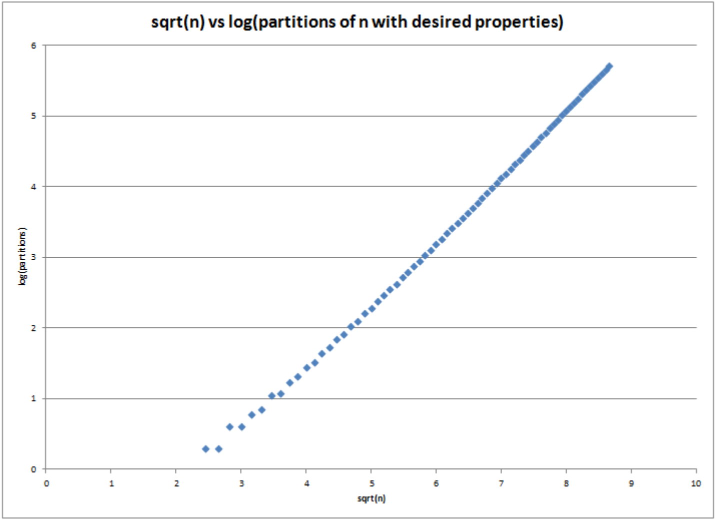

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- We have a recursive formula for determining this; however it is incomplete since we do not know how to find the base case for this formula.

Step 2 - Recurrence Relation

Theorem

Let $e_{k,\{a_1,a_2,\ldots,a_i\}}$ *be the number of orderings with a strategic pile of size* k *and Graph of Possible Merges with the given cycle structure. Then*

$$e_{k,[a_1,a_2,\ldots,a_i]} = \frac{e_{k-1,[a_1,a_2,\ldots,a_i]}k}{k-(a_1+a_2+\ldots+a_i)}$$

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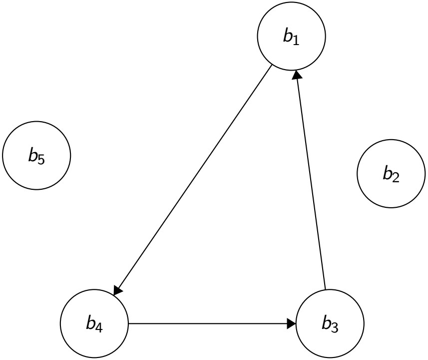

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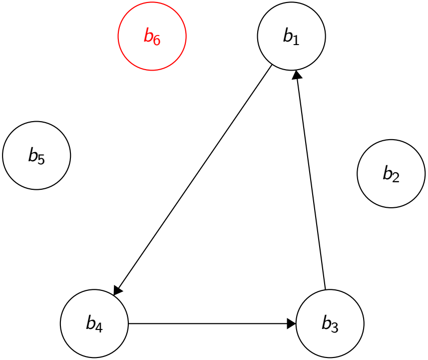

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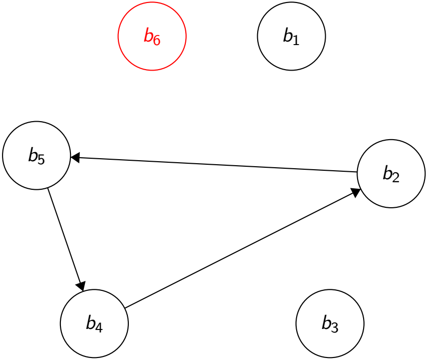

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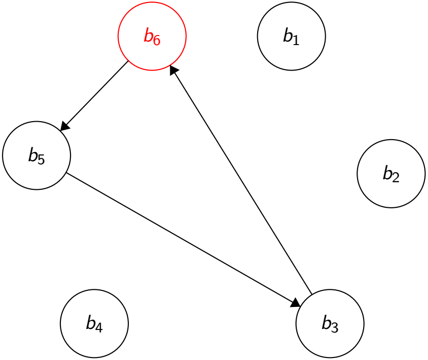

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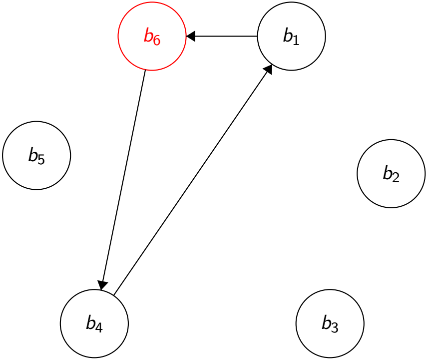

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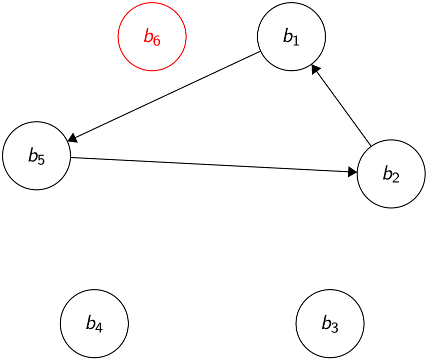

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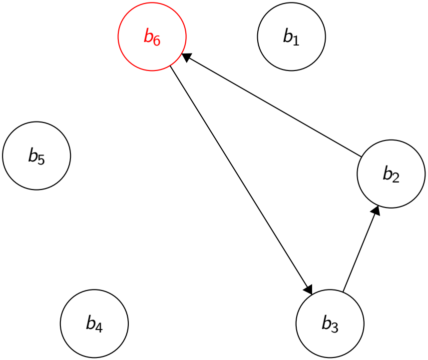

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- Using the recursive formula to find the number of pair orderings that have a given cycle structure runs in no less than polynomial time, as it requires $k - l$ recursive steps.

Proposed Algorithm - Analysis: Step 2

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- For a given cycle structure, calculating this will require 2^i iterations
- Thus, for each cycle structure, this part of the algorithm can be done in $\Theta(e^k)$ steps.
- Since the first part of step 2 can be done in no less than polynomial time, step 2 in it's entirety should take $\Omega(e^k)$ steps for each cycle structure

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- Step 2 will take $\Omega(e^k)$ steps for each cycle structure
- Assuming there are $\Theta(e^{\sqrt{k}})$ cycle structures, our algorithm can be no faster than $\Omega(e^{k^{3/2}})$

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  - Determining the integer partitions of n with all parts of size greater than one and an even number of even parts.
  - Establishing the base case for a recursive formula.
- Construct an alternative merge number algorithm that will run in less than exponential time.

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