Factor Groups of Knots and LOTs

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Definition

Group: $(G, \cdot)$
where G is a set and $\cdot$ is a binary operation. The following properties hold:

- Identity
- Inverses
- Associativity
- Closure

Groups often arise when *symmetry* is present (algebraic symmetry, geometric symmetry, etc)

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Group Presentation:

$$< x_1, x_2, x_3, \ldots | r_1 = 1, r_2 = 1, \cdots >$$

Ex: $< x_1, x_2 | x_1 x_2 = x_2 x_1 >$

Elements are words written using the "alphabet" ${x_1}^{\pm 1}$ and ${x_2}^{\pm 1}$.

Ex: $x_1 x_2 {x_1}^{-1} x_2 x_2 x_1 x_2$

The operation is concatenation.

Ex. $x_1 x_2 x_1 {x_2}^{-1} \cdot x_2 x_1 x_2 = x_1 x_2 x_1 {x_2}^{-1} x_2 x_1 x_2 = x_1 x_2 x_1 x_1 x_2$ $= {x_1}^3 {x_2}^2$.

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How can we tell if the group generated from group presentation is finite or infinite?

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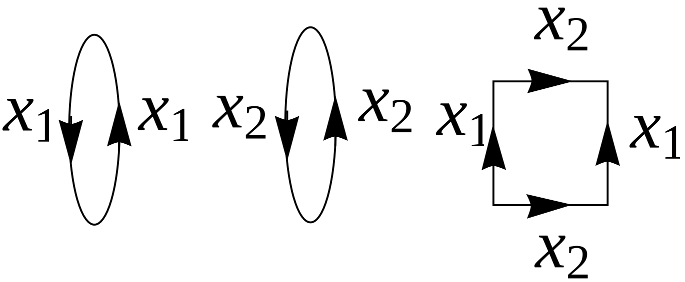
We use Geometry to figure out if a group is finite or infinite.

Definition

Tiles: $< x_1, x_2 | {x_1}^2 = 1, {x_2}^2 = 1, x_1 x_2 = x_2 x_1 >$



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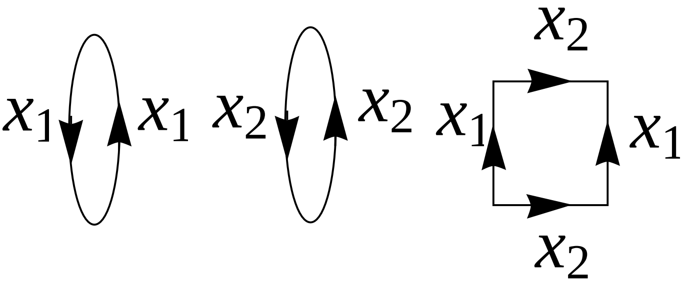

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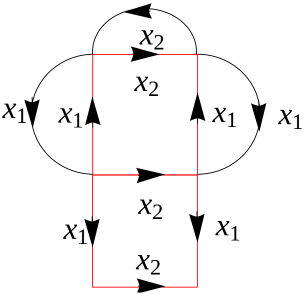
Can we tile a surface with these tiles in an "essential way?"



Theorem (Huebschmann 1979)

If one cannot tile a 2-sphere in an "essential way" using tiles derived from a group presentation, then the group defined by that presentation is either $\mathbb{Z}_n$ or infinite.

Definition (Label Oriented Tree)

A *Label Oriented Tree (LOT)* is a directed, cycle-free graph with vertices labelled $[x_1, ..., x_n]$ and with edges labelled by a vertex[a].

[a] J. Harlander and S. Rosebrock, 2013

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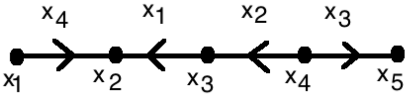

This LOT represents a group with five generators whose relations include $x_1 \cdot x_4 = x_4 \cdot x_2$, $x_3 \cdot x_1 = x_1 \cdot x_2$, $x_4 \cdot x_2 = x_2 \cdot x_3$, $x_4 \cdot x_3 = x_3 \cdot x_5$.

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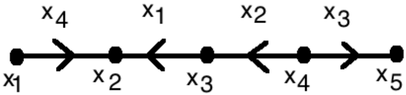

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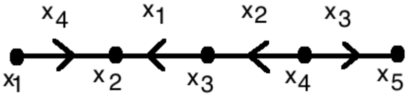

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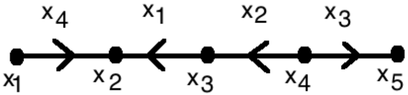

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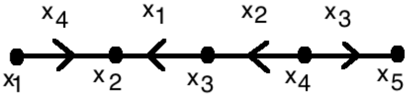

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Note:

- A group with n generators will have $n-1$ relations, as an interval with n vertices will have $n-1$ edges.
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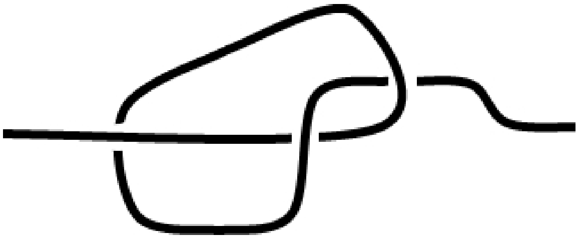

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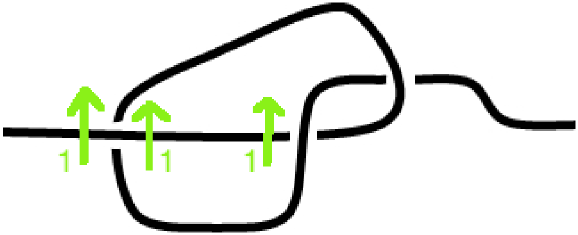

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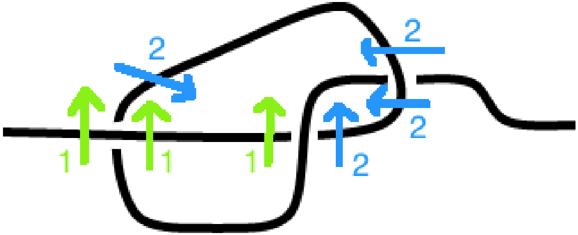

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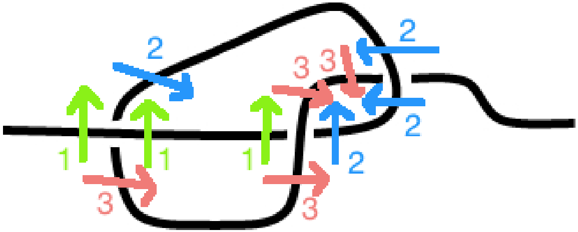

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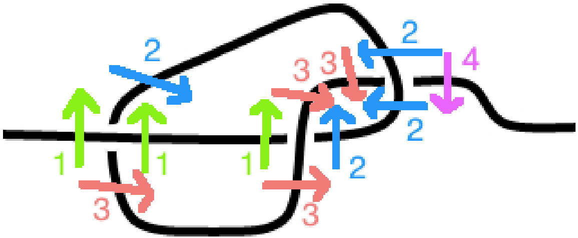
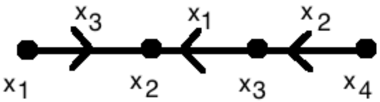


Each intersection of the knot has an associated tile–and therefore, relation–which contributes to our knowledge of the group. The trefoil knot, as drawn previously, is associated with this LOT:



Definition (Q-Series)

Given a LOT T with presentation $P(T)$ we may find $Q_k(T)$ by adding the relation $x_1^k = 1$. The sequence of groups $Q_2(T), Q_3(T), ...$ is called the *Q-Series* of that presentation.

Theorem

If a presentation $P(T)$ comes from a knot, then the Q-Series is a knot invariant.

Proof.

If we look at each of the Reidemeister moves, we can see that they have no affect on the presentation of the knot, and thus do not affect the Q-Series. □

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For the Trefoil Group, $T =< x_1, x_2 \,|\, x_1 \cdot x_2 \cdot x_1 = x_2 \cdot x_1 \cdot x_2 >$

$$\begin{aligned} Q_2 &= S_3 \\ Q_3 &= SL_2 3 \\ Q_4 &= SL_2 3 \ltimes \mathbb{Z}_4 \\ Q_5 &= SL_2 5 \times \mathbb{Z}_5 \\ |Q_{\geq 6}| &= \infty \end{aligned}$$

This result was proved by Coxeter. Our work is an attempt to generalize his results.

Cardinality of the Q-Series

While in the trefoil case we see that $Q_k(T)$ is finite for $k \leq 5$ we expect that generically $Q_k(T)$ will be an infinite group. This is because $Q_k(T)$ has as many relations as it does generators, and generally finite groups have far more relations than generators.

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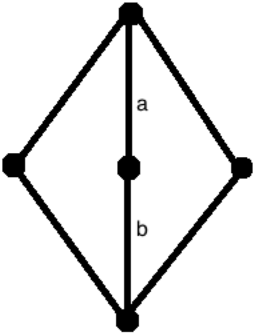
So none of this stuff!



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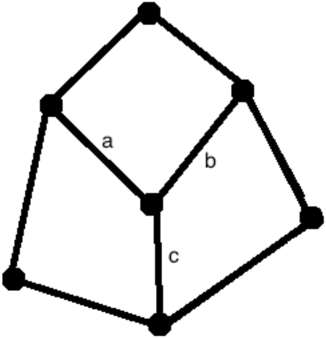

Okay...but what is "appropriately chosen" k supposed to mean?

Theorem

If T is a LOT such that $P(T)$ is a non-positively curved square presentation then $Q_k(T)$ is infinite for appropriately chosen k.

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But in fact we can do better!

If the the theorem holds for odd $k \geq 5$ it follows therefore that the result must hold for all

$$k = p_1^{e_1} \cdots p_n^{e_n}$$

such that $\exists p_i \geq 5$. This is because all prime numbers other than 2 are odd and therefore the theorem must hold for them. We can then consider the map

$$\varphi : Q_k(T) \longrightarrow Q_{p_i}(T)$$

$$x_i \mapsto x_i$$

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All the relations in $Q_k(T)$ are the same as the relations in $Q_{p_i}(T)$ except for one. In $Q_k(T)$ we have $x_1^k = 1$ while in $Q_{p_i}(T)$ we have $x_1^{p_i} = 1$. But since $p_i | k$ it follows that $x_1^k = 1$ in $Q_{p_i}(T)$. Therefore all the relations holding in $Q_k(T)$ also hold in $Q_{p_i}(T)$ making φ a surjective group homomorphism. Since $Q_{p_i}(T)$ is infinte so to is $Q_k(T)$. Thus, this will hold true for all numbers that are not $2^i 3^j$ where $i \geq 0$ and $j = \{0, 1\}$.

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Idea of Proof of Theorem

To any LOT T, we associate a presentation $P(T) =< x_1, \ldots, x_n | \{r_e = 1\} >$, as mentioned before by Renata.

Let this presentation be a non-positively curved square presentation.

Now, to examine the Q Series, we need to consider $P_k(T) =< x_1, \ldots, x_n | \{r_e = 1\}, x_1^k = 1 >$

However, we want to work with $\bar{P}_k(T) =< x_1, \ldots, x_n | \{r_e = 1\}, x_1^k = 1, \ldots, x_n^k = 1 >$

The only difference between $P_k(T)$ and $\bar{P}_k(T)$ is that in the latter we are allowed to use k-gons associated to $x_2^k = 1$ all the way up to $x_n^k = 1$.

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To any LOT T, we associate a presentation $P(T) = < x_1, \ldots, x_n | \{r_e = 1\} >$, as mentioned before by Renata. Let this presentation be a non-positively curved square presentation.
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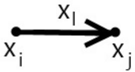
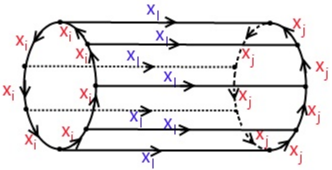
Now, in $\bar{P}_k(T) =< x_1, \ldots, x_n | \{r_e = 1\}, x_1^k = 1, \ldots, x_n^k = 1 >$, consider each of the relations $r_e = 1$.

Each of these relations corresponds to a non-trivial tiling of the 2-sphere in the following way:





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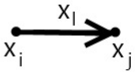

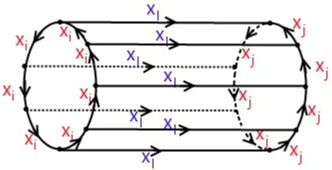

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Given any non-empty spherical diagram over $\bar{P}_k(T)$, it suffices to show that we can reduce the number of tiles in said diagram. I.e. we will show that at least one of the following must occur in any non-empty spherical diagram over $\bar{P}_k(T)$:

1. There exists a mirror image cancellation.
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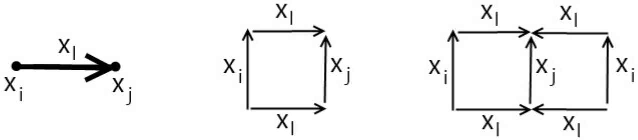
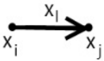
Recall that a mirror image cancellation occurs when we have a tile and its mirror image sharing an edge.
Example:

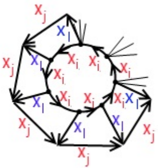


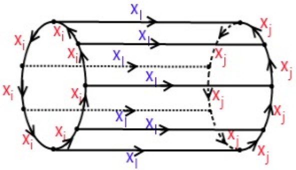
Question

What is an allowable m_e-sphere rewriting?

Best explained by picture example, here is an allowable sphere rewriting:







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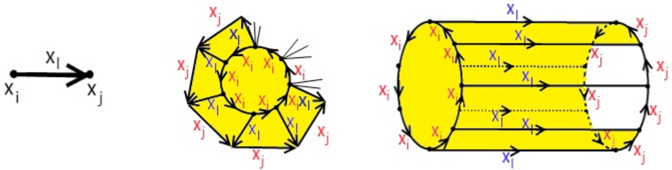

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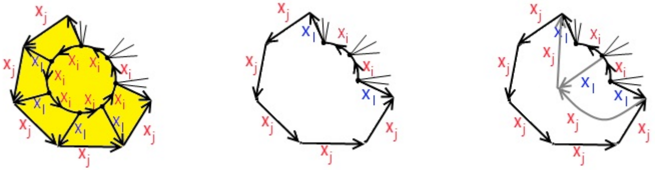
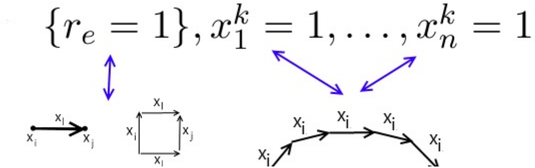


Recall that in $\bar{P}_k(T) =< x_1, \ldots, x_n | \{r_e = 1\}, x_1^k = 1, \ldots, x_n^k = 1 >$, we will be working with the following two types of tiles:



Now, to prove the claim that there must be a mirror image cancellation or an allowable m_e-sphere reduction, we will be working with combinatorial curvature.

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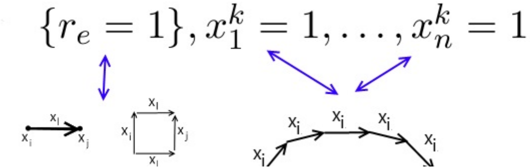

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Given any non-empty spherical diagram over $\bar{P}_k(T)$, it follow that:

Combinatorial Gauss-Bonnet

$\sum \kappa(v) + \sum \kappa(t) = 4\pi.$

Where $\kappa(v)$ is the curvature at a vertex v and $\kappa(t)$ is the curvature of a tile t.

We will begin by assuming that no such mirror image cancellation or allowable m_e-sphere reduction occur in our given spherical diagram.

We will use these facts, indeed, to show that $\sum \kappa(v) + \sum \kappa(t) \leq 0$, which gives us an immediate contradiction to our assumption.

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How do we begin? We want $\sum \kappa(v) + \sum \kappa(t) \leq 0$, so let us start with $\sum \kappa(t)$.

To make things more simple, we will force each tile to have zero face curvature. This immediately gives that $\sum \kappa(t) = 0$.

Easily enough, we accomplish this by letting each tile be a regular, convex n-gon.

Example: Let β be the interior angle.

For our square tiles, $\beta = \frac{\pi}{2}$

And for our k-gons, then $\beta = \frac{(k-2)\pi}{k}$

In particular, if k = 5, then $\beta = \frac{3\pi}{5}$

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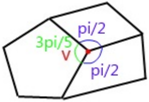
Example of finding $\kappa(v)$: Let one 5-gon and 2 squares meet at v:



Then $\kappa(v) = 2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{3\pi}{5}\right) = \frac{2\pi}{5}$.

The key idea is to classify two types of vertices: either v is touching a k-gon or it is not.

Simply,

$$\sum \kappa(v) = \sum_{\text{k-gon at v}} \kappa(v) + \sum_{\text{no k-gon at v}} \kappa(v)$$

Recall that, because $P(T)$ was a non-positively curved square presentation,

$$\sum_{\text{no k-gon at v}} \kappa(v) \leq 0$$

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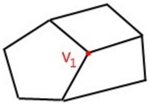
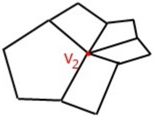
From here, for simplicity of the argument, let k=5.
Looking at the below configurations, notice that the vertex v_1 on the left has positive curvature and the vertex v_2 on the right has negative curvature.





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And the valency 5 vertex v_2 is s.t.
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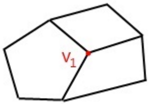

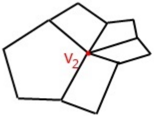

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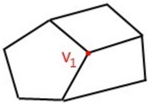

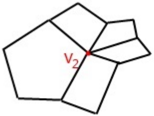

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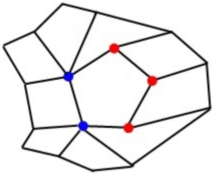

The three red vertices are valency 3 and the two blue vertices are valency 5.
Thus, we sum the curvature around our 5-gon:
$\sum \kappa(v) = 3\left(\frac{2\pi}{5}\right) + 2\left(-\frac{3\pi}{5}\right) = 0$
It takes a significant number of these valency 3 configurations to force the curvature around a k-gon to be positive.
In fact, even having 3 of 5 vertices on our 5-gon having positive curvature was not enough to force the sum of curvature around the 5-gon to be positive.

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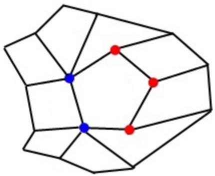

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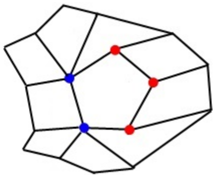

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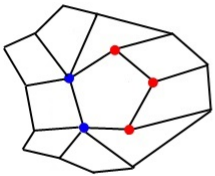

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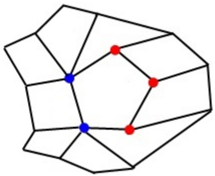
It takes a significant number of these valency 3 configurations to force the curvature around a k-gon to be positive.
In fact, even having 3 of 5 vertices on our 5-gon having positive curvature was not enough to force the sum of curvature around the 5-gon to be positive.

Consider the below tiling:



The three red vertices are valency 3 and the two blue vertices are valency 5.
Thus, we sum the curvature around our 5-gon:

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In fact, even having 3 of 5 vertices on our 5-gon having positive curvature was not enough to force the sum of curvature around the 5-gon to be positive.

Even better, if we happen to find a long enough gallery of these valency 3 configurations, then we are able to apply an allowable m_e-sphere rewriting, as we have seen.

Thus, using these and various other tools, we are able to show that, given any spherical diagram over $\bar{P}_k(T)$, we can always remove tiles.

I.e. Given any spherical diagram, we will be able to arrive at an empty diagram.

Recall the original theorem from Heubschmann: if you cannot tile in an essential way, then the group defined by that presentation is either $\mathbb{Z}_n$ or infinite.

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In fact, our result is close enough to this theorem that we can apply similar topological techniques to find a similar conclusion.

For those topologists in the crowd, we do this by looking at generators of π_2, the set of functions mapping the 2-sphere to the 2-complex defined by our presentation $\bar{P}_k(T)$.

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Questions?