DES over Finite Groups

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1. Introduction
2. How the Project Progressed
3. Main Results
   - DES over Finite Groups
   - Simplified Version of DES over an Elliptic Curve Group
4. Future Work

What is Data Encryption?

- Process of scrambling data so that it can be decoded only by the intended recipient.
- Uses a mathematical algorithm with a key to encode a file into a form that cannot be read.
- Used to protect government records, military secrets and majority of businesses.

DES: The First Encryption Standard

- Symmetric block data encryption technique.
- Published by the National Bureau of Standards in 1975.
- Accepted as a federal encryption standard in the U.S. in 1977, and later internationally.

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Asymmetric encryption uses two keys, one public and one private. The public key is used to encrypt a message, and the private key is used to decrypt it.

Use of the Data Encryption Standard

- Electronic financial transactions
- Secure data communications
- Protection of passwords or PINs against unauthorized access.

"Whoever you are, I can guarantee that many times in your life, the security of your data was protected by DES."

Bruce Schneier, 2004

Overview of DES

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- Plaintext space: $\mathcal{P} = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (64 copies of $\mathbb{Z}_2$).
- Ciphertext space: $\mathcal{C} = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (64 copies of $\mathbb{Z}_2$)
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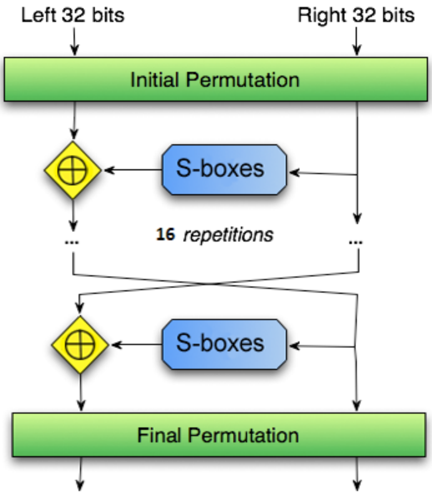
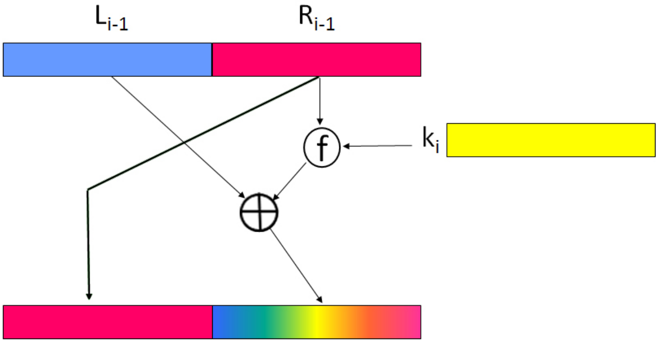
- $|\mathcal{P}| = |\mathcal{C}| = 2^{64}$.
- For each key $K \in \mathcal{K}$, the encryption function $e_K : \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ has a very technical, but clear, description.
- Each $e_K \in \mathcal{E}$ is a permutation of the set $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (64 copies of $\mathbb{Z}_2$).
- Each decryption $d_K \in \mathcal{D}$ is a permutation of the set $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (64 copies of $\mathbb{Z}_2$).
- $|\mathcal{E}| = |\mathcal{D}| = 2^{56}$.

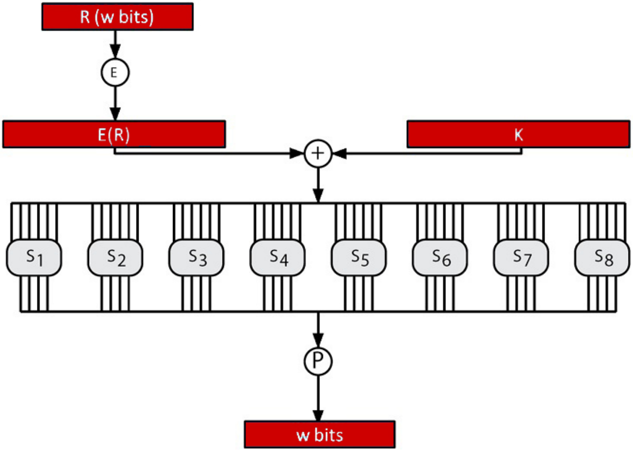
Outline of DES



Feistel Round



Feistel Function



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There is either a conclusive proof or statistical evidence that gives an answer to all these questions for DES over $\mathbb{Z}_2$ only.

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Initial Work

- We studied two simplified version of DES:
  - B-DES encrypts over $\mathbb{Z}_2$. It uses six Feistel rounds, a 12-bit message, and a 9-bit key.[2]
  - S-DES encrypts over $\mathbb{Z}_2$. It uses two Feistel rounds, an 8-bit message, and a 10-bit key.[3]

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- We created a program in Maple to analyze S-DES over the groups $U(5)$ and $U(8)$. Then, using a combination of Coppersmith method and known algebraic results, we proved the following:

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Theorem

B-DES and S-DES over U(5) and U(8) are not groups.

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More Results

We extended our existing Maple software and proved the following:

Theorem

S-DES does not form a group over $\mathbb{Z}_n$ (for certain S-boxes) when n is divisible by 2, 3, 5, 7, or 11.

Theorem

Let G be a finite group. Then S-DES does not form a group over $\mathbb{Z}_n \times G$ (for certain S-boxes) when n is divisible by 2, 3, 5, 7, or 11.

DES-Like Functions

Definition

Suppose that G is a group and $k > 1$. A **DES-like function** on G^{2k} is a transformation δ_f on G^{2k} determined by a function $f : G^k \to G^k$ as follows:

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$$\delta_f(\bar{x}, \bar{y}) = (\bar{y}, \bar{x} \oplus f(\bar{y})),$$

where $\bar{x}, \bar{y} \in G^k$ and $\oplus$ is the group operation of G.

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Theorem

The set of all δ_f (on fixed G^{2k}) does not form a group under composition.

S-DES

Definition

S-DES is a function $S_k = \theta \circ \delta_{f_2} \circ \delta_{f_1}$, where θ is a right-left swap and δ_{f_1} and δ_{f_2} are the functions corresponding to the first and second key schedule (for some fixed S-box).

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We can rewrite this using the definition of δ_{f_i}:

$$S(\bar{x}, \bar{y}) = (\bar{y} \oplus f_2(\bar{x} \oplus f_1(\bar{y})), \bar{x} \oplus f_1(\bar{y})).$$

S-DES with Constant S-boxes

Theorem

Let G be a finite group. Consider a version of S-DES over G for which each entry in every S-box is some fixed element $g \in G$. Then the following are true:

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Let G be a finite group. Consider a version of S-DES over G for which each entry in every S-box is some fixed element $g \in G$. Then the following are true:

1. *S-DES (over all possible keys) does not form a group under functional composition.*
2. *The group generated by S-DES is a cyclic group of order* $\mathrm{lcm}(2, |g|)$.

S-DES is Not a Group

Theorem

S-DES over a group of order ≥ 2 is not a group for any S-box.

DES-Like Encryptions

Definition

A **DES-like encryption** with m rounds is a function $D = \theta \circ \delta_m \circ \ldots \circ \delta_2 \circ \delta_1$, where $\delta_{f_i}(\bar{x}, \bar{y}) = (\bar{y}, \bar{x} \oplus f_i(\bar{y}))$ and $\theta(\bar{x}, \bar{y}) = (\bar{y}, \bar{x})$.

DES-Like Encryptions with Constant S-boxes

Theorem

Let D_k be a family of DES-like encryptions with $m \geq 2$ rounds, different keys, and constant S-boxes that return $g \in G$. If m is even or $|g| > 2$, then the following are true:

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2. *The group generated by these encryptions is a cyclic group of order $|g|$ if m is odd, or $\operatorname{lcm}(2, |g|)$ if m is even.*

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Theorem

A DES-like encryption over a finite group with constant S-boxes is not faithful.

DES is Not a Group

Theorem

A DES-like encryption over a finite group of order ≥ 2 does not form a group unless both of the following hold:

1. *The DES-like encryption has an odd number of Feistel rounds.*
2. *Every element of every S-box is the identity.*

Groups Generated by DES-Like Encryptions

In 1983 S. Even and O. Goldreich showed that DES-Like functions over $\mathbb{Z}_2$ can generate the alternating group.[4]

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In 1983 S. Even and O. Goldreich showed that DES-Like functions over $\mathbb{Z}_2$ can generate the alternating group.[4]

Theorem

Let G be a group such that $|G|$ is odd and $|G|^k \bmod 4 \equiv 3$, where k is an odd integer. Then the group of permutations generated by the DES-like function $\delta_f : G^{2k} \to G^{2k}$ contains odd permutations.

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Corollary

Let G be a group such that $|G|$ is odd and $|G|^k \bmod 4 \equiv 3$, where k is an odd integer. Then the group of permutations generated by the DES-like function $\delta_f : G^{2k} \to G^{2k}$ is either $S_{|G|^{2k}}$ or a subgroup H of $S_{|G|^{2k}}$ such that half of the elements in H are even permutations.

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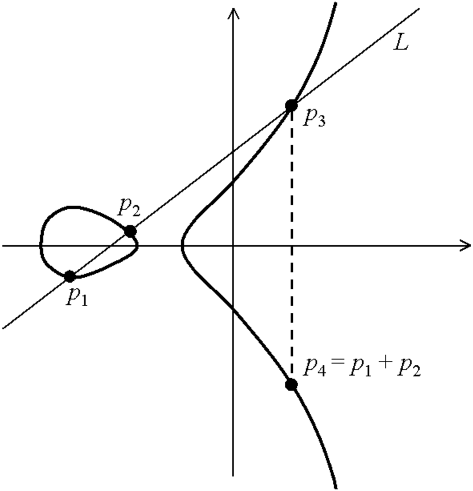
Elliptic Curves

An **Elliptic Curve** is given by an equation of the form

$$y^2 = x^3 + Ax + B,$$

where A and B are constants and the discriminant $\Delta = 4A^3 + 27B^2$ is nonzero.

Addition on an Elliptic Curve



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Theorem

The addition of points on $E = \{(x, y) : y^2 = x^3 + Ax + B\} \cup \{\infty\}$ *satisfies the following properties:*

1. *Commutativity:* $P_1 + P_2 = P_2 + P_1$ *for all* P_1, P_2 *on* E.
2. *Existence of Identity:* $P + \infty = P$ *for all points* P *on* E.
3. *Existence of Inverses: Given a point* P *on* E, *there exists a point* P' *on* E *such that* $P + P' = \infty$.
4. *Associativity:* $(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$ *for all* P_1, P_2, P_3 *on* E.

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4. *Associativity:* $(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$ *for all* P_1, P_2, P_3 *on* E.

In other words, the points on E *form an additive abelian group with* ∞ *as the identity element.*

Elliptic Curves Over Finite Fields

Theorem

Let E be an elliptic curve over a finite field. Then $E \cong \mathbb{Z}_n$ or $E \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ for some integers $n_1, n_2 \geq 1$ such that $n_1 | n_2$.

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Theorem

Let E be an elliptic curve over a finite field. If DES is not a group over $\mathbb{Z}_n$ for any n, then there exist S-boxes such that DES is not a group over the finite field.

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- Encrypts over the elliptic curve $y^2 \equiv x^3 + 4 \mod 7$

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- Message length: 10 bits
- Key length: 12 bits
- Expander function: $[r_5 r_1 r_2 r_3 r_4 r_2 r_3 r_4 r_5 r_1]$
- Key schedule: $K_1 = [k_2 k_{10} k_8 k_9 k_1 k_5 k_{11} k_7 k_6 k_5]$ and $K_2 = [k_2 k_3 k_1 k_6 k_{10} k_9 k_5 k_{11} k_{12} k_4]$

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- Two S-boxes are 9×9 with 2-nit outputs. Each row and column is a permutation of the elements of $\mathbb{Z}_3 \times \mathbb{Z}_3$.
- One S-box is 3×3 with a 1-nit output. Each row and column is a permutation of the elements of $\mathbb{Z}_3$.

Future Work

Solve the following open problems related to the security of DES over finite groups:

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Improve the security of E-DES in the following ways:

- Construct S-boxes that satisfy specific properties.
- Design a new key schedule using the Discrete Log Problem.

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