CYCLE TRANSFORMATIONS AND CDR — A UNIFYING CONCEPT

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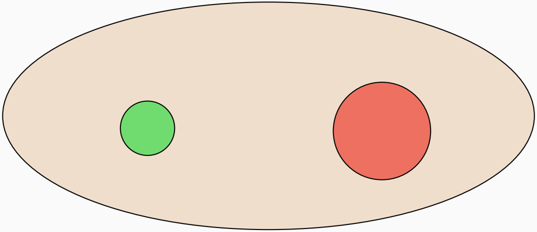
Overview

INTRODUCTION

Motivation: Ciliates

http://www.microbehunter.com/forum/photographs-and-videos/ciliates-1/

What are Ciliates?



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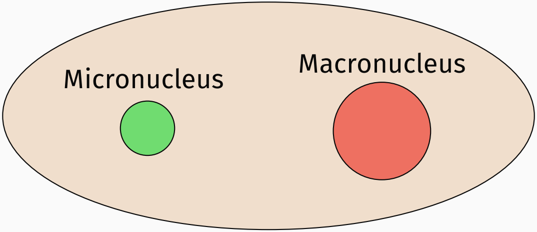
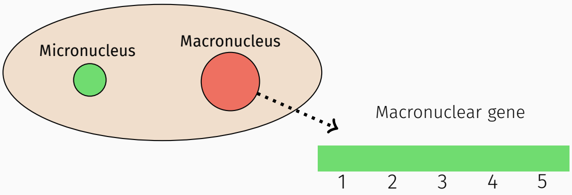


DNA Sorting via Context Directed Reversals



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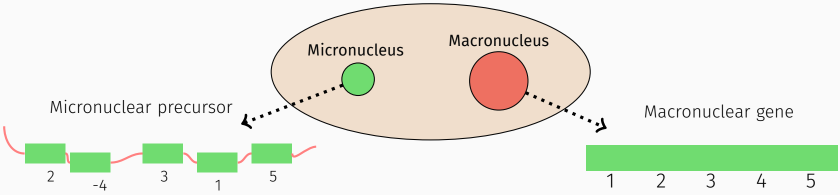

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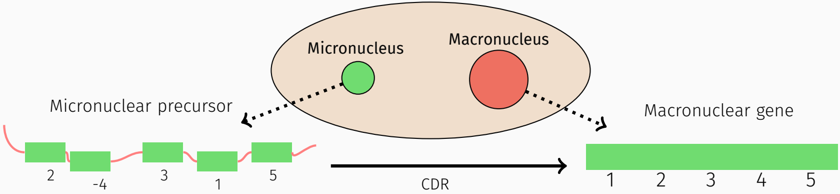

Representing DNA as Signed Permutations

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$$\pi = \left[1_{(1,2)} \quad {}_{(1,2)}2_{(2,3)} \quad {}_{(2,3)}3_{(3,4)} \quad {}_{(3,4)}4_{(4,5)} \quad {}_{(4,5)}5 \right]$$

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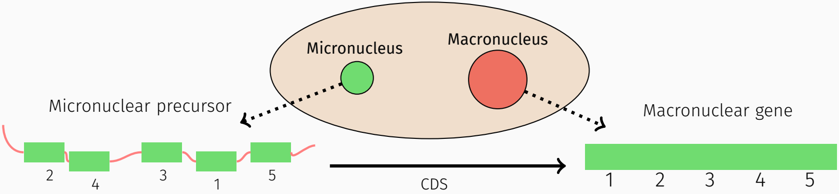

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Fixed Points

If there are no more -p,p pointer parings (no further reversals to be made) and/or there are no more p,q pointer pairings (no further swaps to me made), then the signed permutation is sorted, or at a **fixed point**.

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CDR (CDS) Fixed Point

A signed permutation that has **no valid CDR (CDS) moves** available.

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- Otherwise, π is **non-sortable**.

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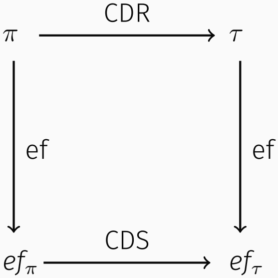
As you can see, CDR in the extended form is a constrained version of CDS.



ALGEBRA AND SORTING

Terminology

Definition

For $i \in \{1, \ldots, n\}$ and π a signed permutation of degree n, we say that i is **negative in** π if i appears to the left of 0 in ef_π. Otherwise, we say i is **positive in** π.

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For $i, j \in \{1, \ldots, n\}$ and $\pi \in B_n$, we say that there is an i, j **sign change** in π if exactly one of the two is negative in π. (Equivalently, we say this if the two appear on opposite sides of 0 in ef_π.)

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We study how sign changes and moves in π relate to those in π^{-1}.

Inverses and Moves

Theorem

Let π be an arbitrary signed permutation. There is an $(x, x+1)$ pointer move in π^{-1} if, and only if, there is a $\pi(x), \pi(x+1)$ sign change in π.

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We study how for π_1, π_2, moves and sign changes in both affect available moves in $\pi_2 \circ \pi_1$.

Theorem

Let π_1, π_2 *be arbitrary signed permutations of degree n. Let x be an element in* $\{1, \ldots, n\}$*. Let* $i, j \in \{1, \ldots, n\}$ *be such that* $\pi_2(i) = \pm x$*,* $\pi_2(j) = \pm(x + 1)$*. Then we have the following:*

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1. *Suppose there is a CDR move on the $(x, x+1)$ pointer in π_2, then there is a CDR move on the $(x, x+1)$ pointer in $\pi_2 \circ \pi_1$ if and only if there is no i, j sign change in π_1.*

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2. *Suppose there is no CDR move on the $(x, x+1)$ pointer in π_2, then there is a CDR move on the $(x, x+1)$ pointer in $\pi_2 \circ \pi_1$ if and only if there is an i, j sign change in π_1.*

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We have reduced the problem of reachability to one of sortability when considering whether π can reach a fixed point.

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Let π be an arbitrary signed permutation. Let f be a CDR fixed point. Then f is CDR reachable from π if and only if $f^{-1}\pi$ is CDR sortable.

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This proof follows from the fact that the inverses of fixed points are fixed points and composition by fixed points does not affect which moves are available in the composition. □

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Note that this collapses the problem of reachability to a problem of sortability.

CYCLE TRANSFORMATIONS

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$$D_\pi = \begin{pmatrix} 0 & 6 & \color{red}{5} & \color{red}{-4} & \color{red}{1} \end{pmatrix} \begin{pmatrix} -1 & -2 & 3 & -6 & -7 \end{pmatrix} \begin{pmatrix} 2 & -5 \end{pmatrix} \begin{pmatrix} -3 & 4 \end{pmatrix}$$

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$$D_\pi = \begin{pmatrix} 0 & 6 & {\color{red}5} & {\color{red}-4} & {\color{red}1} \end{pmatrix} \begin{pmatrix} -1 & -2 & 3 & -6 & -7 \end{pmatrix} \begin{pmatrix} 2 & -5 \end{pmatrix} \begin{pmatrix} -3 & 4 \end{pmatrix}$$

The strategic stack of this example for $\pi = \begin{bmatrix} 2 & -4 & 6 & 1 & -3 & 5 \end{bmatrix}$ is the set: $\{5, -4, 1\}$, appearing in the first cycle of D_π after $n = 6$.

Removing Elements from the Cycle Transformation

Theorem

For a given element i of a signed permutation π, doing a CDR move on the $(i, i+1)$ pointer results in i and $-(i+1)$ being removed from their respective cycles in D_π.

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Theorem

The number of elements that can be removed from the strategic stack through a given CDR move is either 0 or 1.

The Dual

Definition: Dual

The **dual** of an element i, denoted $d(i)$, is the element $-(i+1)$.

Dual Results

Dual Mapping Theorem

For a given signed permutation, π, an element i maps to x in D_π if and only if $d(x)$ maps to $d(i)$.

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Dual Separation Theorem

The element i and its dual **can not** appear in the same cycle of D_π.

The Dual Cycle

Definition: Dual Cycle

For a cycle transformation D_π, the **dual cycle** of a cycle ϕ contained in D_π, where $\phi = (a_1 \ldots a_k)$, is the cycle of the duals of each element of ϕ, denoted, $d(\phi)$, where $d(\phi) = (d(a_k) \ldots d(a_1))$.

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A cycle $\phi = (a_1 \dots a_k)$ is contained in D_π if, and only if, the **dual cycle** of ϕ, $d(\phi) = (d(a_k) \dots d(a_1))$, is contained in D_π.

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$$d(\phi) = (d(0) \;\; d(5) \;\; d(-8) \;\; d(2) \;\; d(-5) \;\; d(3))$$

$$=(-1 \; -6 \;\; 7 \; -3 \;\; 4 \;\; -4).$$

Let α_π be the cycle in D_π containing 0 and let β_π be the cycle in D_π containing n.

Cycle Transformation Sortability Criteria

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(c) *If $\alpha_\pi \neq \beta_\pi$ and $\alpha_\pi \neq d(\beta_\pi)$, then π is mixed sortable.*

(d) *If there exists in D_π a cycle of nonnegative elements which appear in π (including zero), then π is not CDR sortable.*

FUTURE DIRECTIONS

- Expand Reachability Theorem to arbitrary pairs of signed permutations

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- Use results to study games and determine winning strategies

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