Anomalous Primes and Elliptic Carmichael Numbers

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Boise State University

2016 REU CAD Symposium

Outline

Introduction

Problems in Cryptography

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- Encryption

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- Primality testing

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Elliptic curves can tackle both of these problems!

Elliptic Curves

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Definition

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Usually K is $\mathbb{F}_p$ for a prime p or $\mathbb{F}_q$ for q a prime power.

Elliptic Curves are Groups

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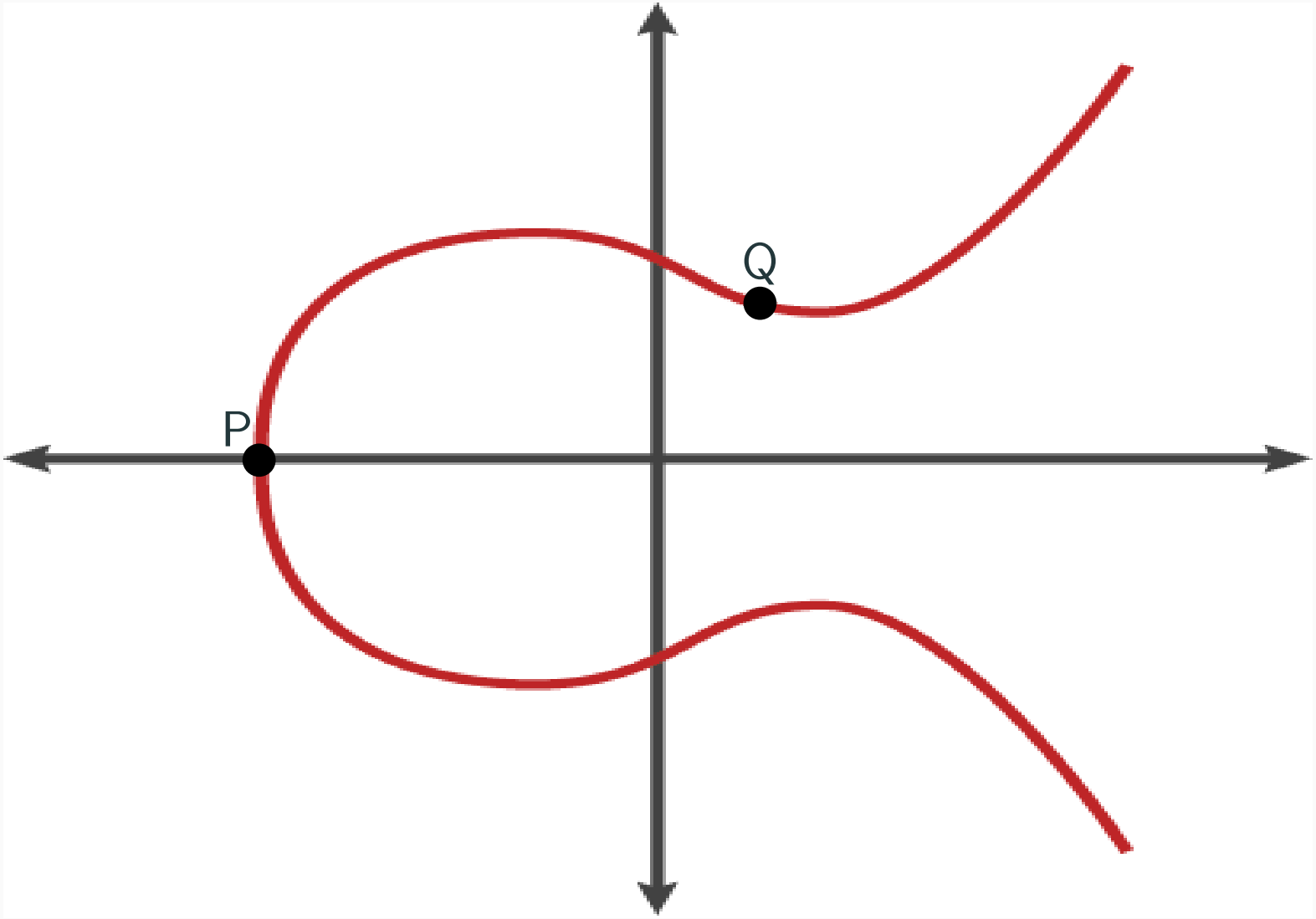

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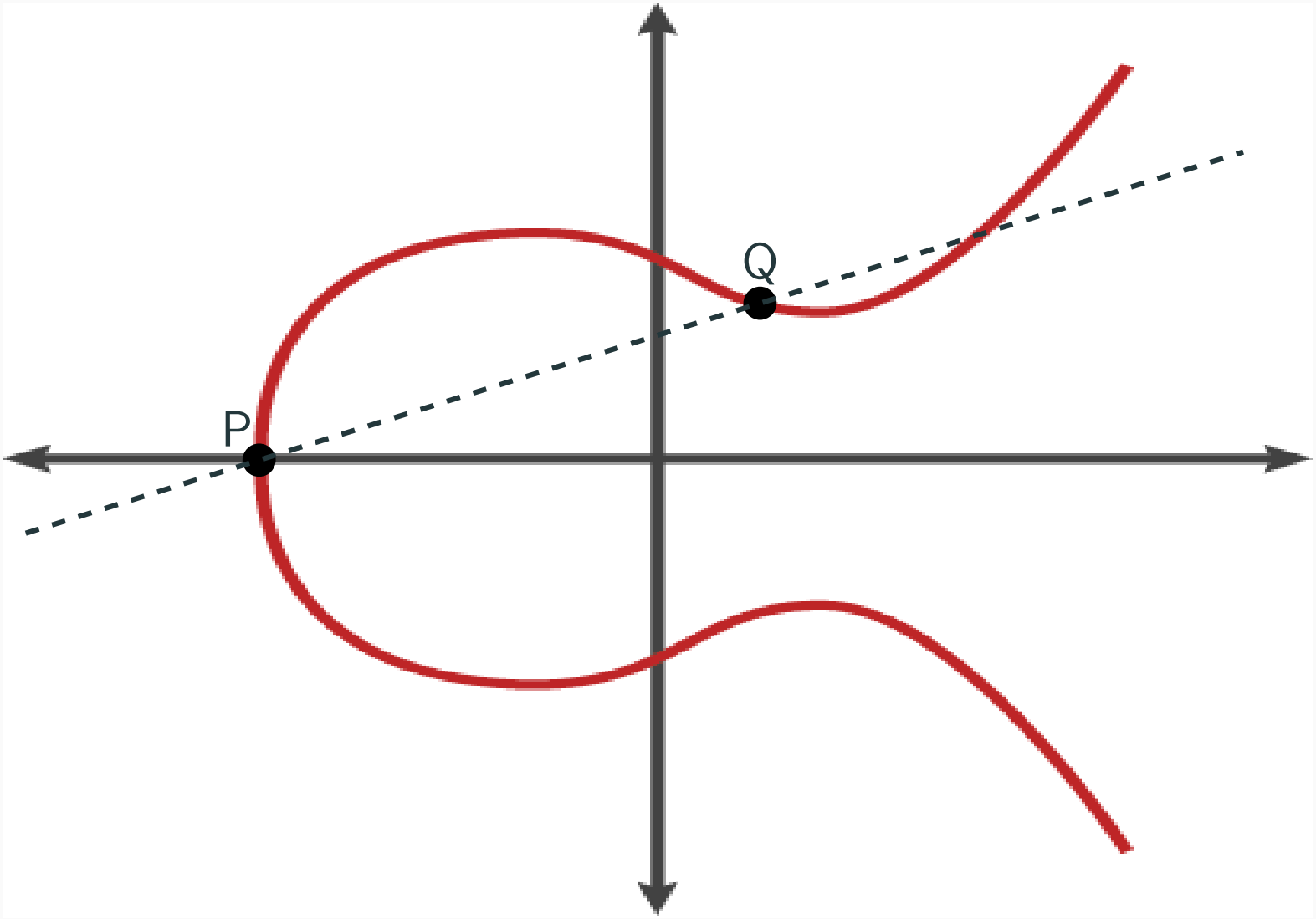

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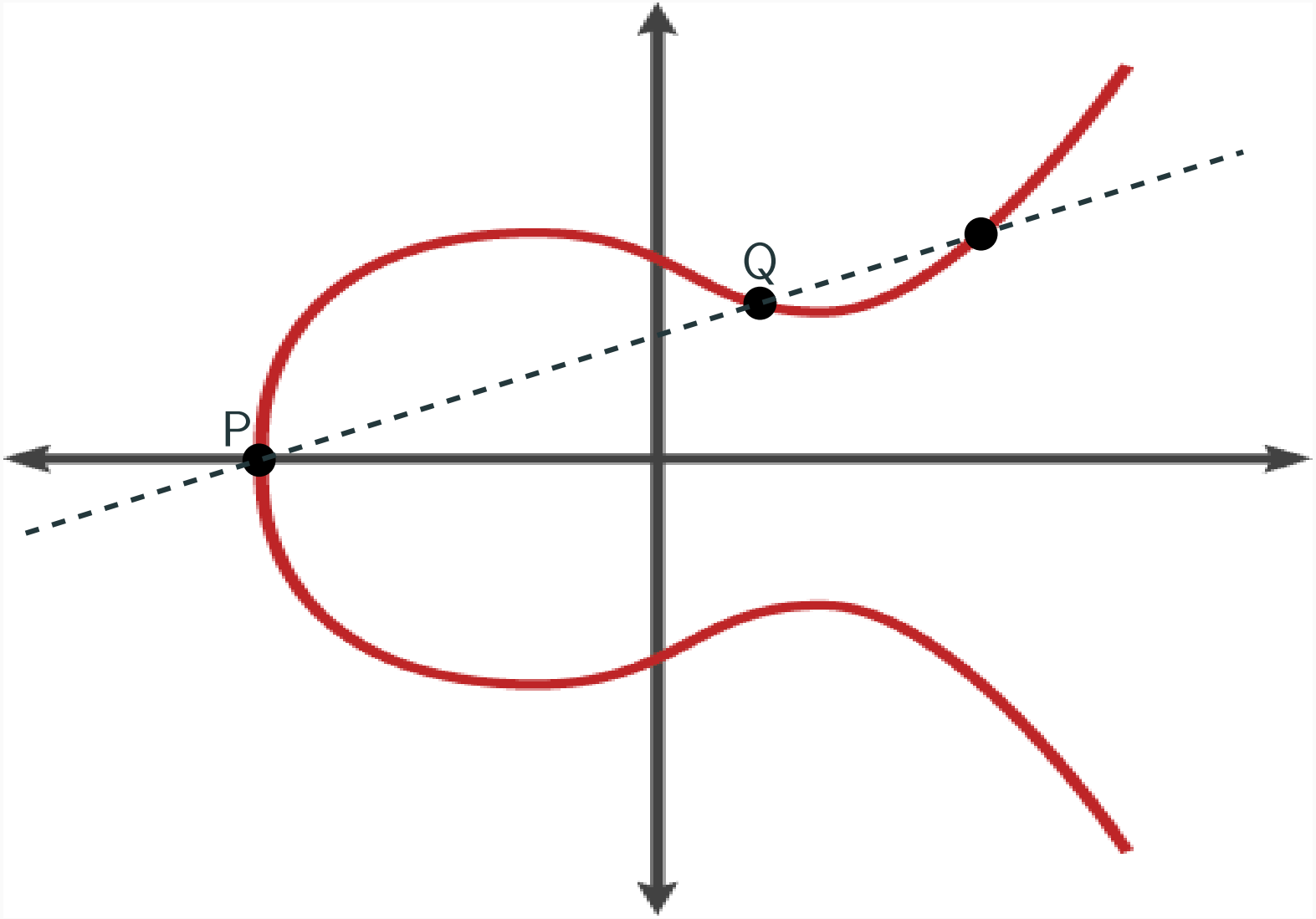

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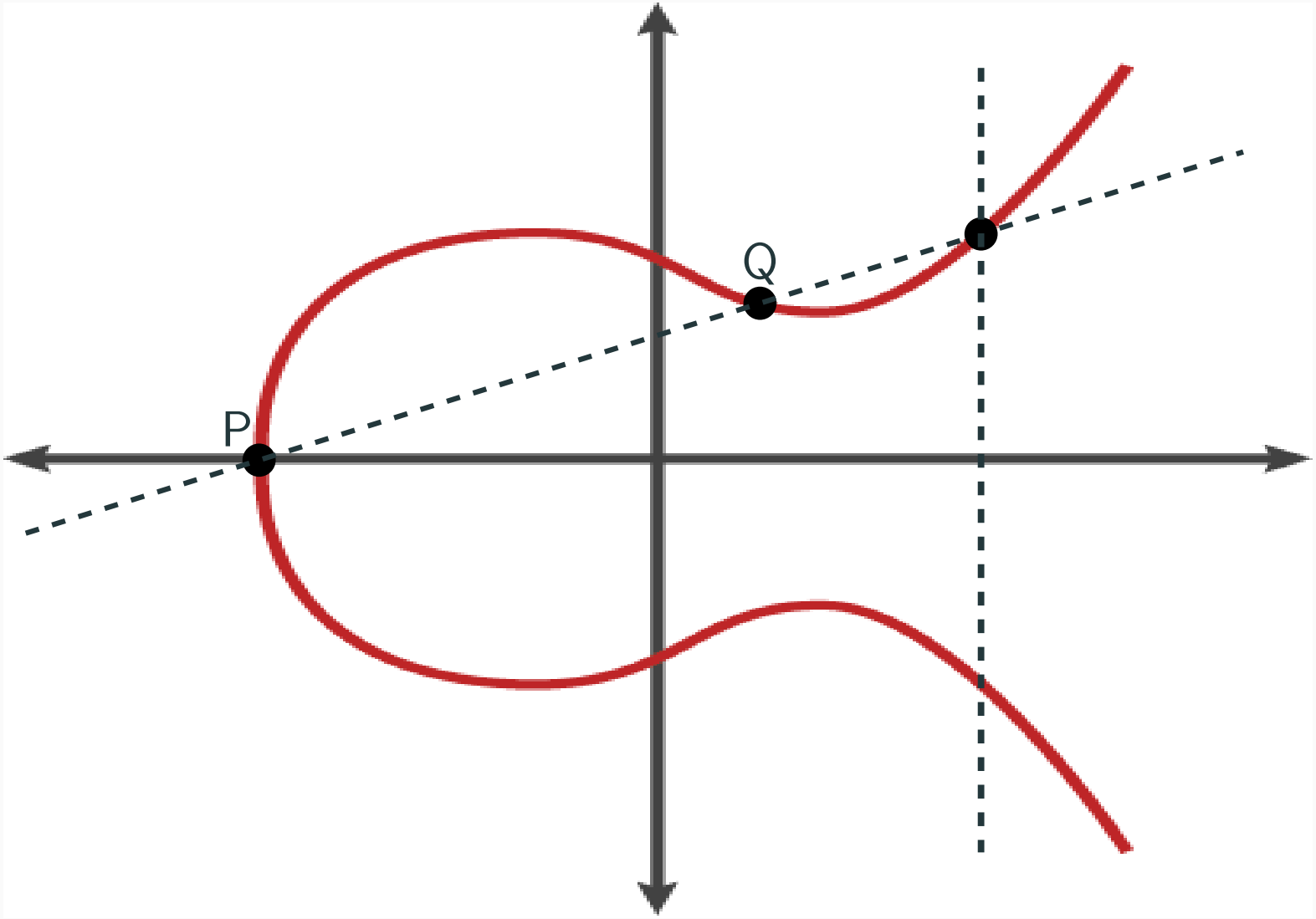

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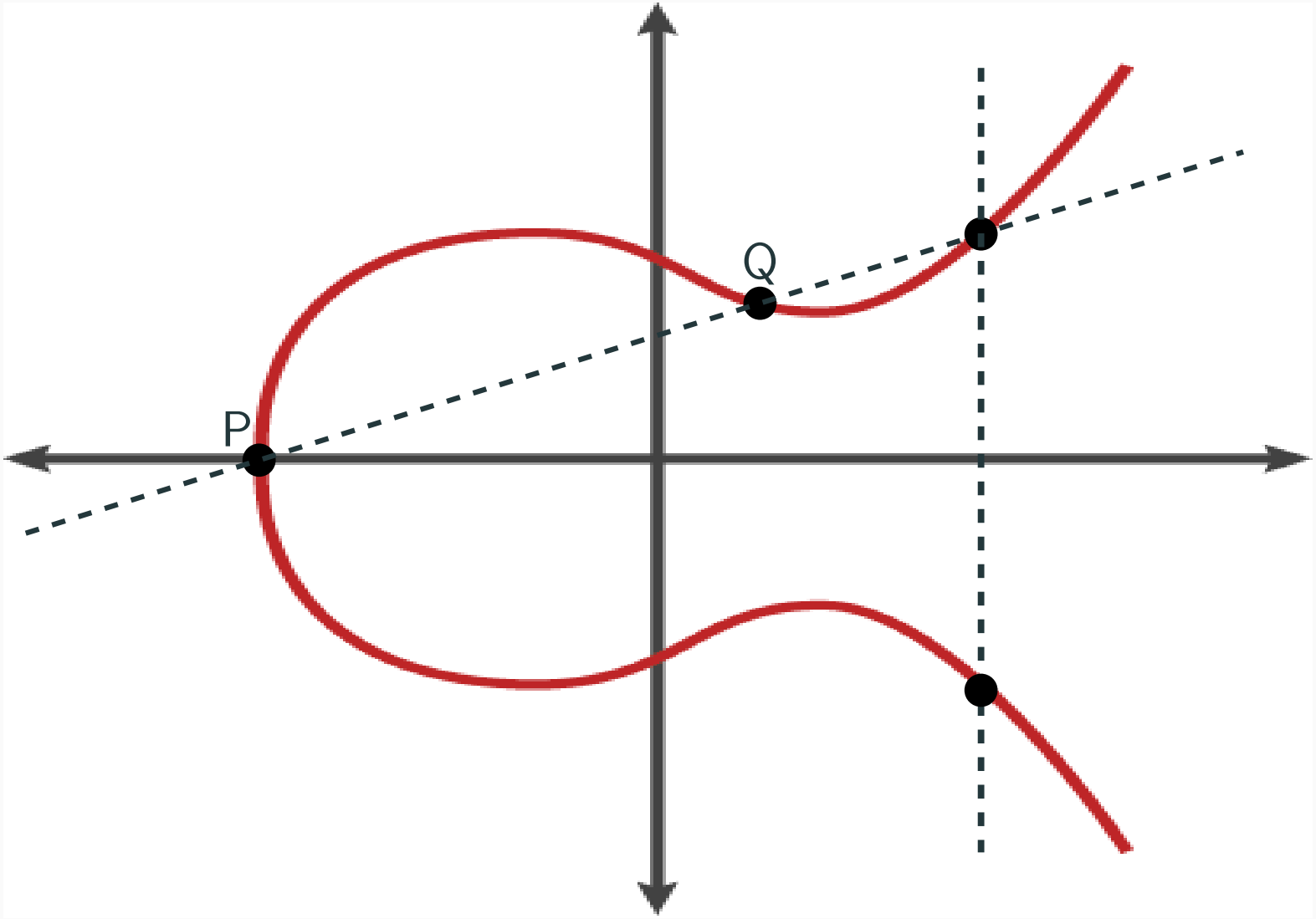

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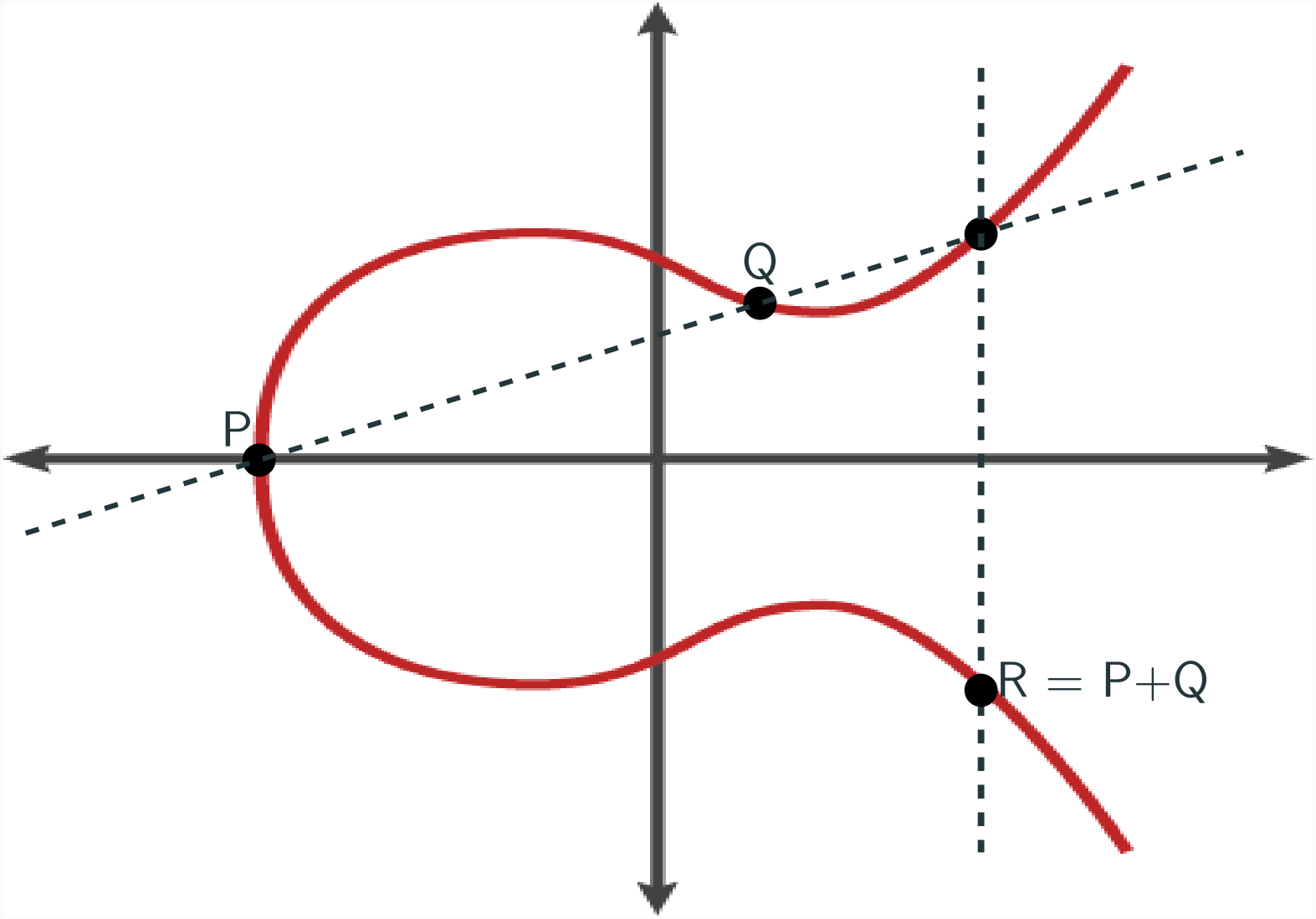

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Definition

The **j-invariant** of an elliptic curve $E : y^2 = x^3 + ax + b$ is

$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}$$

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Theorem (Hasse)

Let q be a prime power and $E(\mathbb{F}_q)$ an elliptic curve with trace a_q. Then $|a_q| \leq 2\sqrt{q}$.

Elliptic Curve Orders

Curves with $j \neq 0, 1728$

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Theorem

If $j_0 \neq 0, 1728$, then there is an integer $a_q(j_0)$ such that for all curves $E(\mathbb{F}_q)$ with j-invariant equal to j_0

$$\#E(\mathbb{F}_q) \in \{q + 1 \pm a_q(j_0)\}$$

Moreover, each order is realized by some curve.

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Let $E(\mathbb{F}_q)$ be an elliptic curve with $j = 0$. If $q \equiv 1 \pmod 3$, then E takes on one of 6 possible orders. Moreover, there are curves with $j = 0$ that realize each order.

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Theorem

If $q = p^r$ where $p \equiv 2 \pmod 3$ and r even, then if $E(\mathbb{F}_q)$ has $j = 0$, then

$$\#E(\mathbb{F}_q) \in \{p + 1 \pm 2p^{r/2}, p + 1 \pm p^{r/2}\}$$

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$$\#E(\mathbb{F}_q) \in \{p + 1, p + 1 \pm 2p^{r/2}\}$$

Summary

	$j = 0$		$j = 1728$		$j \notin \{0, 1728\}$
	$p \equiv 1 \pmod 3$	$p \equiv 2 \pmod 3$	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$	
$\mathbb{F}_p$	6 orders	1 order	4 orders	1 order	2 orders
F_{p^r} (r odd)	6 orders	1 order	4 orders	1 order	2 orders
F_{p^r} (r even)	6 orders	4 orders	4 orders	3 orders	2 orders

Bachet Anomalous Primes and Bachet Anomalous Numbers

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p is called a **Bachet anomalous prime** if there exists an elliptic curve E of the form $y^2 = x^3 + B$ such that $\#E(\mathbb{F}_p) = p$.

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- Elliptic curves of the form $y^2 = x^3 + B$ are exactly the curves with j-invariant 0.
    - These curves are known as **Bachet elliptic curves**.

Bachet Anomalous Primes and Centered Hexagonal Numbers

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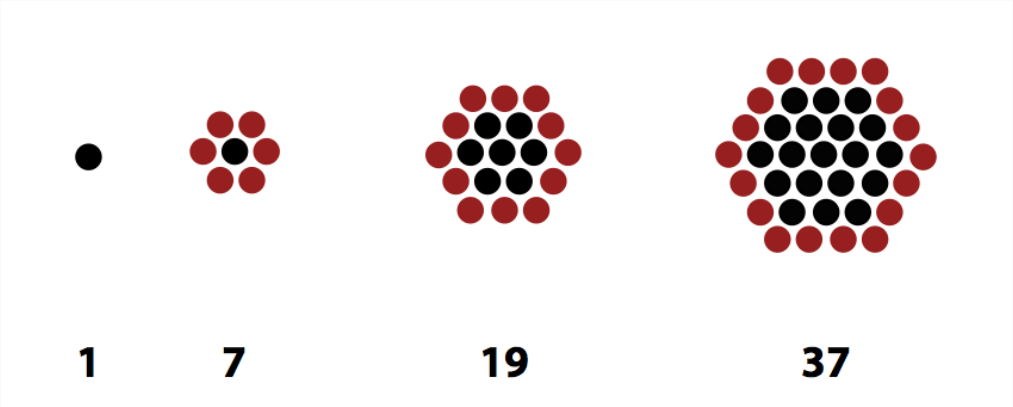

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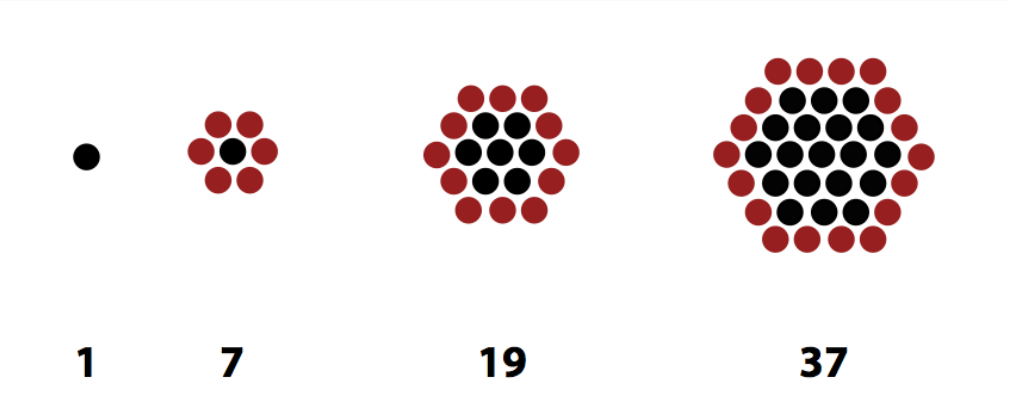

Theorem

A prime number p is a Bachet anomalous prime if and only if it is a centered hexagonal number.

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Definition (Generalizing Bachet anomalous primes)

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Let A, B, C, x, y, and z be positive integers, with $x, y, z > 2$. If A, B, and C are relatively prime, then $A^x + B^y = C^z$ has no integer solutions (A, B, C).

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Theorem

Let $p^r = (n+1)^3 - n^3$ be a prime power centered hexagonal number. When $r \in \{1, 2\}$, p^r is always a Bachet anomalous number. Assuming T-Z conjecture, there are no Bachet anomalous numbers p^r when $r \geq 3$.

Type I Elliptic Korselt Numbers

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- Just like regular Carmichael numbers, these are numbers that pass a primality test despite being composite.

Type I Elliptic Korselt Numbers

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Definition

For an elliptic curve E, a positive integer n is called a **Type I elliptic Korselt Number (EK-I Number)** if it has at least two distinct prime factors and for each prime factor p,

$$p + 1 - a_p \mid n + 1 - a_n \text{ and}$$

$$\operatorname{ord}_p(a_n - 1) \geq \operatorname{ord}_p(n) - \begin{cases} 1 & \text{if } a_p \not\equiv 1 \pmod{p} \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem

If n is an EK-I number for an elliptic curve E, then n is an elliptic Carmichael number for E [S, Proposition 11].

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Every product of distinct anomalous primes for an elliptic curve E is a Type I elliptic Korselt number for E.

Relevant Inclusions

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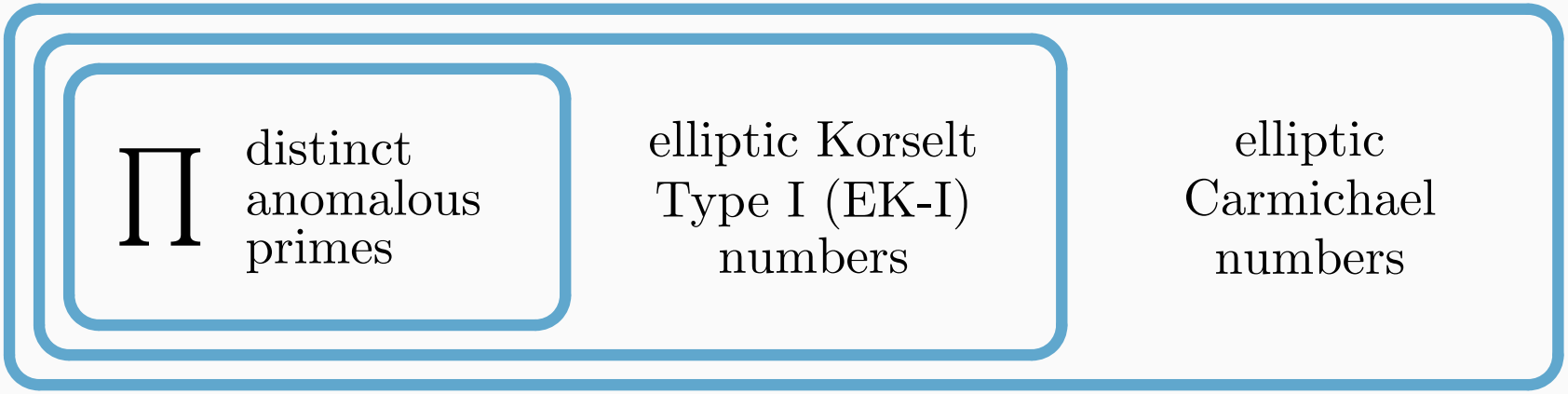

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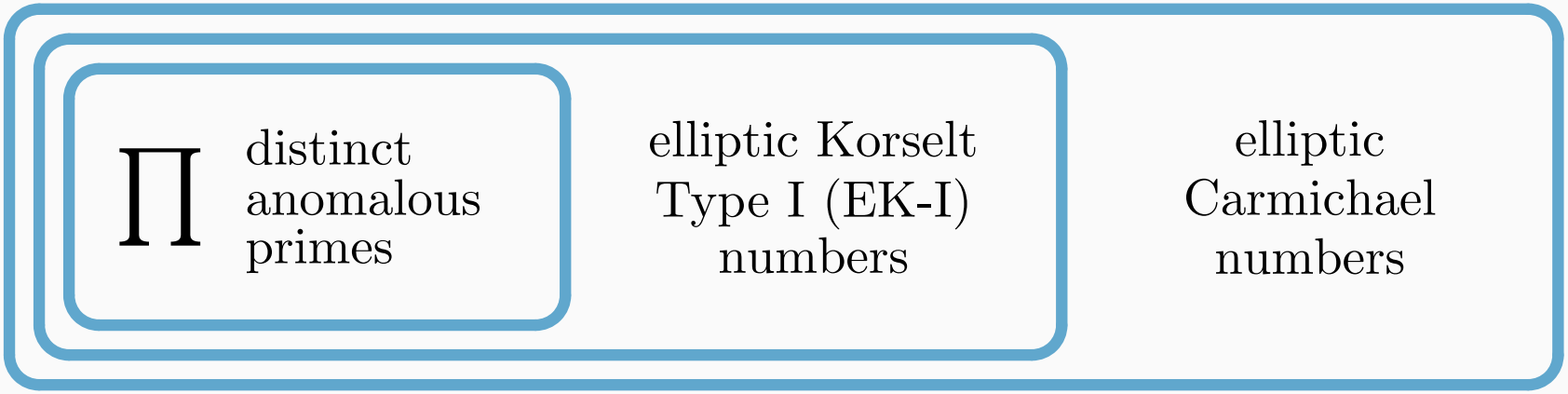

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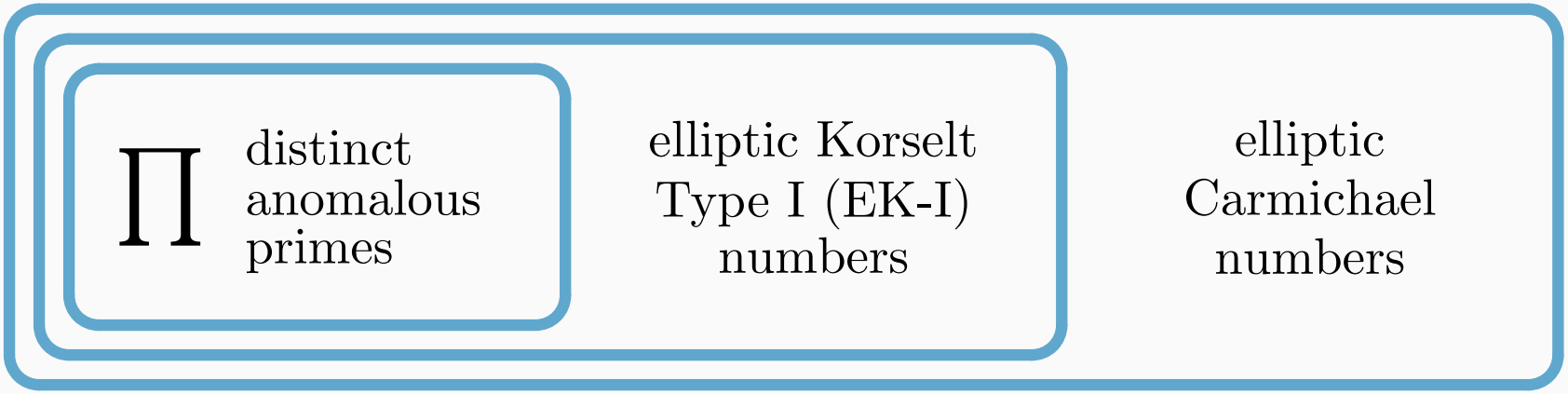

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- We made progress on that question for the inner inclusion.

A Restriction on EK-I Numbers

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- For $m = 2$ this gives us a circumstance under which E must be anomalous over p_1 and p_2 (when $16 < p_1 < \sqrt{p_2}16$), correcting Silverman's result.
- The drawback: these conditions are "rarely" satisfied.

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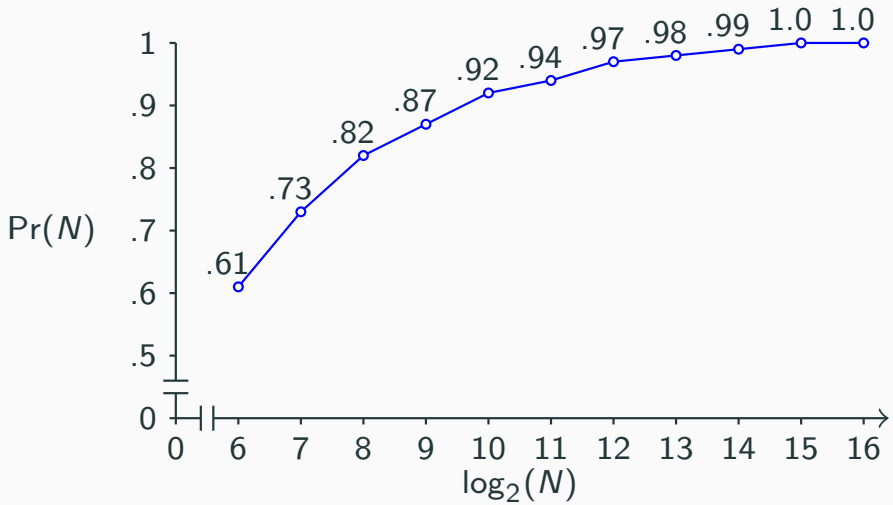

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- Future work: potentially extend theorem to m primes.
- If such an extension is plausible, then anomalous primes are the cornerstone to understanding EK-I numbers.

References

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