Computability and Complexity in Elliptic Curves and Cryptography

An Algorithm for Finding Elliptic Curves of Prime Order over $\mathbb{F}_p$

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Collaborators

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The use of elliptic curves in cryptography necessitates the ability to construct curves of prime order, which are believed to be the most secure in cryptographic applications due to the Silver - Pohlig - Hellman algorithm for computing the discrete logarithm.

Our Motivation

Conjecture

[abc] When E has CM, the probability that its order $\#E$ is prime is inversely proportional to $\lg p$.

[a] Koblitz, Primality of the Number of Points on an Elliptic Curve over a Finite Field, *Pacific Journal of Mathematics*, Vol. 131, No. 1 (1988), pp.157-165. Conjecture B.

[b] Silverman and Stange, Amicable Pairs and Aliquot Cycles in Elliptic Curves, *Experimental Mathematics*, Vol. 20, No. 3 (2011), pp.329-357. Conjecture 3(b).

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Definition

An *elliptic pair* is an ordered pair of prime numbers (p, q) such that the order of $E : y^2 = x^3 + k \mod p$ for some $k \not\equiv 0 \mod p$ is q.

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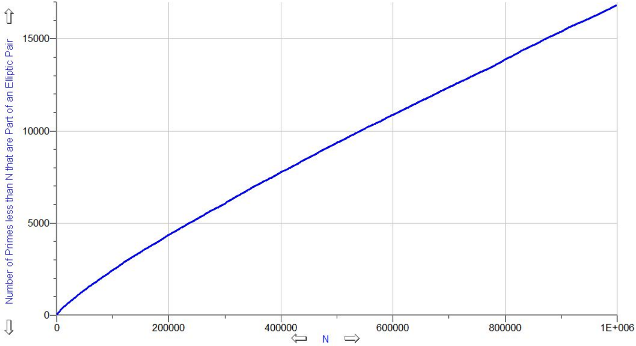
Since $\pi(N) \sim \frac{N}{\lg N}$, this conjecture is equivalent to saying that the number of elliptic pairs is proportional to $\frac{N}{\lg^2 N}$

The Number of Elliptic Pairs



The number of elliptic pairs less than N is approximately

$$A \frac{N}{\lg^2 N},$$

with experiment indicating $A \approx 0.6056$.

Outline of the Algorithm

Algorithm

Input: A number of bits, N, that the prime p should be
Output: An elliptic curve E over $\mathbb{F}_p$ such that $\#E$ is prime

1. Generate prime value p of desired size using Miller-Rabin primality test.
2. Find primitive root g modulo p.
3. Compute the order of $E : y^2 = x^3 + g$.
4. Test for primality using Miller-Rabin. If composite, start over and generate a new value for p.

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Sieving using small primes and finding values for p of the form $p = x^2 + 3y^2$ can reduce the number of values we have to try down to polynomial in $\lg N$.

This reduces the overall complexity to $\tilde{O}(N^3)$

The order of E when $j = 1728$

Theorem

[a] Let $p > 3$ be an odd prime and let $k \not\equiv 0 \mod p$. Let $N_p = \#E(\mathbb{F}_p)$, where E is the elliptic curve $y^2 = x^3 - kx$.

1. If $p \equiv 3 \mod 4$, then $N_p = p + 1$.
2. If $p \equiv 1 \mod 4$, write $p = a^2 + b^2$, where a, b are integers with b even and $a + b \equiv 1 \mod 4$. Then

$$N_p = \begin{cases} p + 1 - 2a & \text{if } k \text{ is a fourth power mod } p \\ p + 1 + 2a & \text{if } k \text{ is a QR but not a 4th power mod } p \\ p + 1 \pm 2b & \text{if } k \text{ is a QNR mod } p. \end{cases}$$

[a] Washington, *Elliptic Curves: Number Theory and Cryptography, 2nd Ed.*, **Taylor & Francis Group**, (2008). pp.115-123. This theorem is a classical result, dating back to Gauss.

Here, QR and QNR denote quadratic residue and quadratic non-residue, respectively.

The order of E when $j = 0$

Theorem

Let $p > 3$ be an odd prime and let $k \not\equiv 0 \mod p$. Let $N_p = \#E(\mathbb{F}_p)$, where E is the elliptic curve $y^2 = x^3 + k$.

1. If $p \equiv 2 \mod 3$, then $N_p = p + 1$.
2. If $p \equiv 1 \mod 3$, write $p = a^2 + 3b^2$,[a] where a, b are integers with b positive and $a \equiv -1 \mod 3$.

[a] Most previous authors formulate this theorem using $p = a^2 - ab + b^2$ instead.

Here, CR denotes cubic residue.

$E : y^2 = x^3 + k$

We compute $\#E(\mathbb{F}_p) = p + 1 - \pi - \bar{\pi}$, where $\pi = 0$ if $p \equiv 2 \mod 3$, and $\pi = -\chi_6(k)^{-1} J(\chi_2, \chi_3)$ otherwise.

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Corollary

Let $p \equiv 1 \mod 3$ be a prime and let $k \not\equiv 0 \mod p$. If $\#E(\mathbb{F}_p)$ is prime, then k is neither a QR nor a CR, except in the case where $p = 7$ and $k = 4$.

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Note that in the case $j = 1728$, $\#E(\mathbb{F}_p)$ is always even, so it is not prime.

Computation of a and b

The Smith-Cornacchia Algorithm is used to compute integers a and b such that

$$a^2 + db^2 = m$$

for given integers d and m.

1. First, we find a value r_0 such that $r_0^2 \equiv -d \mod m$. If no such value exists, then there is no solution.
2. Next, we perform the Euclidean algorithm, computing $r_1 \equiv m \mod r_0$, etc.
3. We terminate the algorithm when we reach a value of $r_i < \sqrt{m}$. Then either $a = r_i$, $b = \sqrt{\frac{m - r_i^2}{d}}$ is a solution, or no solution exists.

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Since we are working with $m = p$ is prime, the first step can be achieved rapidly using the algorithm of Tonelli and Shanks.

Comparison to Other Algorithms

Bröker and Stevenhagen[1] also suggested an algorithm for generating an elliptic curve of prime order which runs in $\tilde{O}(N^3)$, but it is more difficult to implement.

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Our Algorithm	Their Algorithm
p is modulus of curve	p is order of curve
Most time-consuming step involves finding N-bit primes	Most time-consuming step involves finding appropriate $D \equiv 5 \mod 8$
Test primality of order at end of algorithm	Try to compute Hilbert class polynomial P_D and find elliptic curve at end

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Future Directions

There are some incomplete results for dealing with the general CM case.[2][3] These are worthy of further investigation and generalization.

[2] Cohen, *Number Theory, Volume 1: Tools and Diophantine Equations*, **Springer**, (2007). pp.565-572. Section 8.5.2.

[3] Rajwade, *Arithmetic on curves with complex multiplication by* $\sqrt{-2}$, Proclamations of the Cambridge Philosophical Society, Vol. 64. (1968), pp.659-672.

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It is unknown whether curves of the form $E : y^2 = x^3 + k$ are less suitable than others for cryptographic purposes, due to their simple form (although there is currently no evidence for this). The other CM curves should be incorporated into the algorithm.

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We still need a better analysis of the time complexity for finding primes of a given size. It is also probable that there exists a faster algorithm than the one we found.

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[a] If 2 is not a CR, then if (p, q) is an elliptic pair, so is (q, p).

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Can we use these results to prove that the number of elliptic pairs less than N is proportional to $\frac{N}{\lg^2 N}$?

Acknowledgements

Funding for this project is provided by the National Science Foundation (DMS 1062857) and by Boise State University.