Computability and Complexity in Elliptic Curves and Cryptography

SubBytes, MixColumns, and 1-round AES

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Collaborators

In collaboration with
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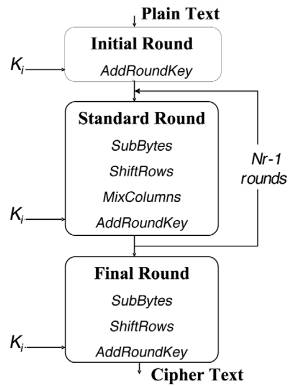
AES Overview



[1]

[1] J. Daemen and V. Rijmen, *AES submission document on Rijndael*, Version 2, (1999)

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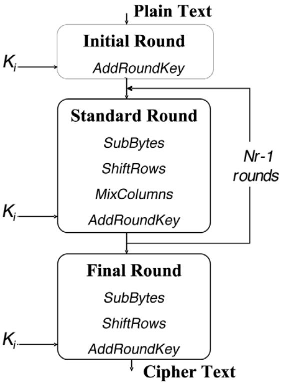

[1]

My focus: Parity of SubBytes and MixColumns, followed by 1-round AES-like functions.

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SubBytes

Recall that the plaintext can be represented as a matrix.

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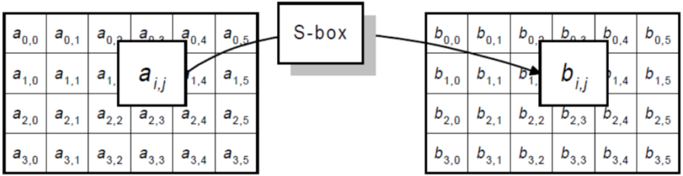
Definition

The function $\lambda : M_{m,n}((GF(p^r)) \rightarrow M_{m,n}((GF(p^r))$ is called a *SubBytes function* if it is the parallel application of mn bijective S-box-mappings $\lambda_{ij} : GF(p^r) \rightarrow GF(p^r)$ defined by $\lambda(a) = b$ if and only if $b_{ij} = \lambda_{ij}(a_{ij})$ for all $0 \leq i < m, 0 \leq j < n$.

[2] 

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SubBytes

SubBytes puts each element in the matrix through an 'S-box' λ_{ij} given by

$$\lambda_{ij}(x) = ax^{-1} + b$$

where

- a is a degree $r - 1$ polynomial
- x^{-1} is the inverse of the element over $GF(p^r)$
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The S-box is usually implemented as a lookup table.

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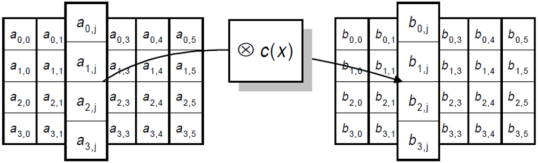
- m and n are both odd, and
- Each individual S-box λ_{ij} is odd:
  - $p \equiv_4 3$, r is odd, and $(p^r - 1)/\left|\langle a \rangle\right|$ is odd, OR
  - either $p \equiv_4 1$ or r is even, and $(p^r - 1)/\left|\langle a \rangle\right|$ is even, OR
  - $p = 2$ and $r > 1$.

MixColumns

Definition

ρ is a *MixColumns function* if it is an invertible linear transformation over $M_{m,n}(GF(p^r))$, ie, there is an invertible matrix $D \in M_{m,m}(GF(p^r))$ such that
$\rho(x) = Dx, \quad \forall x \in M_{m,n}(GF(p^r))$.



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Lemma

The MixColumns function ρ is an odd permutation if and only if p and n are both odd, and $(p^{rm} - 1)/\left|\langle c \rangle\right|$ is odd, where c is the fixed polynomial of the MixColumns function ρ.

1-round AES Functions

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Definition

For any $k \in \mathcal{K}$, a *generalized 1-round AES permutation* $T[k] : GF(p^r)^{mn} \to GF(p^r)^{mn}$ is a permutation of the form $T[k] = \sigma[k] \circ \rho \circ \pi \circ \lambda$ where

λ	is a SubBytes function,
π	is a ShiftRows function,
ρ	is a MixColumns function, and
$\sigma[k]$	is the AddRoundKey function with key k.

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- For the set of 1-round AES functions we write $\tau := \{T[k] | k \in \mathcal{K}\}$.
- For the group generated by τ we write $G_\tau := \langle \tau \rangle$.

1-round AES Functions

Theorem

One-round classical AES generates the alternating group.[a]

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Open problem: what group do general 1-round AES permutations generate?

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Open problem: what group do general 1-round AES permutations generate?

We suspect it is always the alternating or symmetric group.

Tools: Transitivity

Let G be a permutation group over a set X.

Definition

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Lemma

The group generated by 1-round AES functions over $M_{m,n}(GF(2^r))$ is transitive $\forall m, n, r$.[a]

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G is *ℓ-transitive* if $\forall (a_i, b_i)$, a_i, $b_i \in X$, $1 \leq i \leq \ell$, such that $a_j \neq a_k$ and $b_j \neq b_k$ when $j \neq k$, $\exists \sigma \in G$ such that $\sigma(a_i) = b_i$.

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Theorem

If G is a 2-transitive group of degree N containing a p-cycle where p is prime and $N/2 < p \leq N - 3$, then G is the alternating or symmetric group. [a]

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Definition

A set $B = \{b_1, \ldots, b_k\} \subset X$ is a *block* under G if $\forall \sigma \in G$, either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$.

Definition

G is *primitive* if $\nexists$ any nontrivial block of X under G.

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Theorem

If G is a primitive group of degree N containing an M-cycle where $2 \leq M \leq (N-M)!$, then G is the alternating or symmetric group. [a]

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Acknowledgements

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