Cycle Transformations and CDR — A Unifying Concept

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Motivation: Ciliates

Ciliates are single-celled organisms that rely on sorting for survival. Research suggests that context directed swaps (CDS) and context directed reversals (CDR) sort ciliate micronuclear DNA during the construction of a new macronucleus. [2]

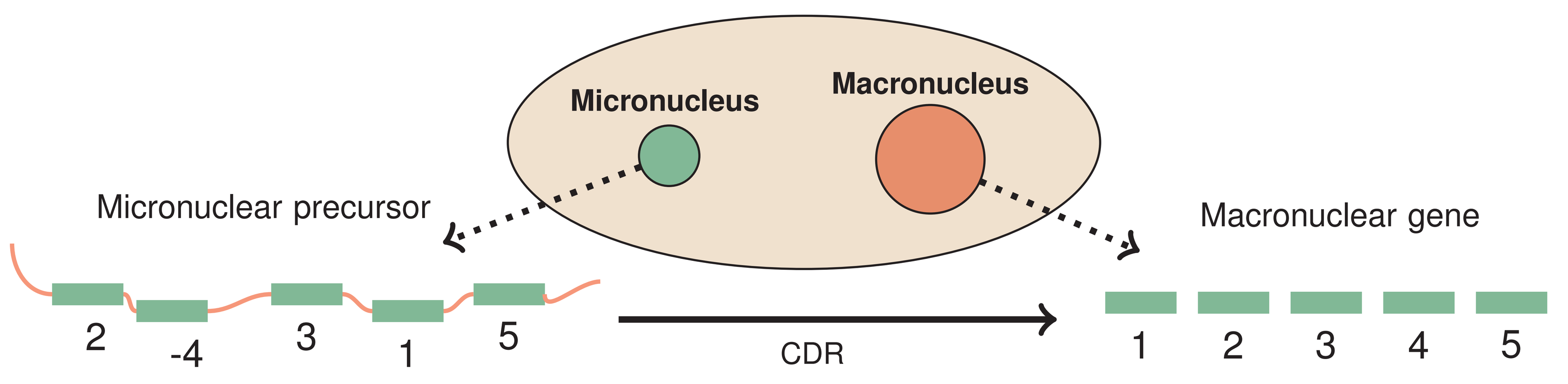

Figure 1: CDR as used by ciliates

The focus of our research is to investigate and analyze how outcomes of CDR sorting can be predicted. We are developing a criterion for reachability via solving a sortability problem for permutations.

Context Directed Operations

Context Directed Reversals

To each entry i of a signed permutation π assign the left pointer $\langle i-1, i\rangle$ and the right pointer $\langle i, i+1\rangle$. For a negative entry, pointers switch sides and become negative. Only a segment of π bounded by a pointer p and $-p$ may be reversed by a CDR move.

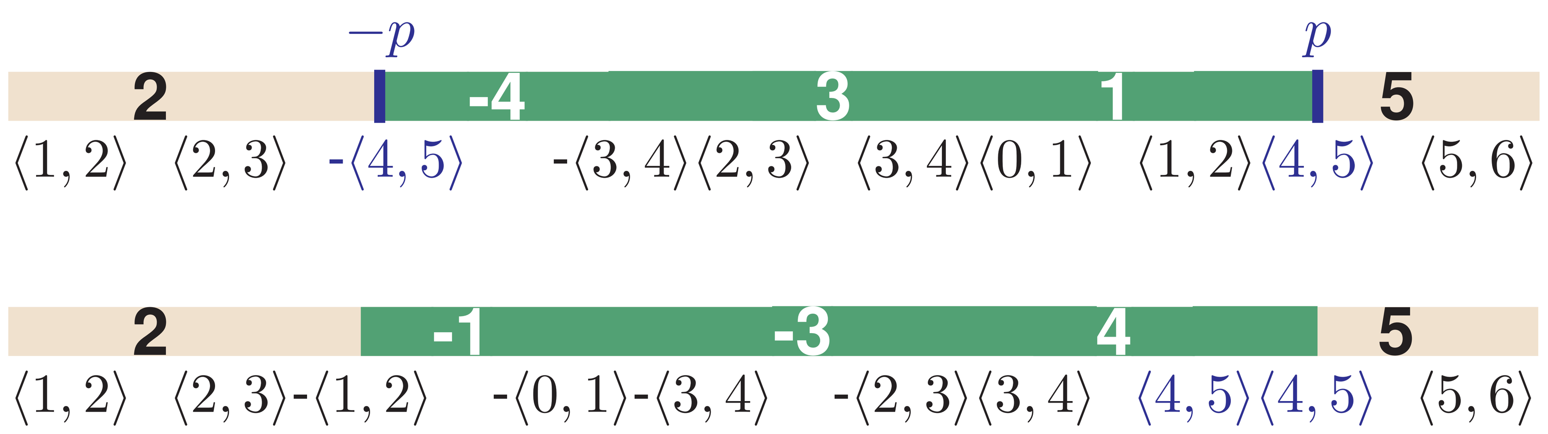

Figure 2: CDR on the signed permutation [2 -4 3 1 5]

Context Directed Swaps

Only two disjoint segments of π, each bounded by pointers p and q in the same order, may be swapped by a CDS move.

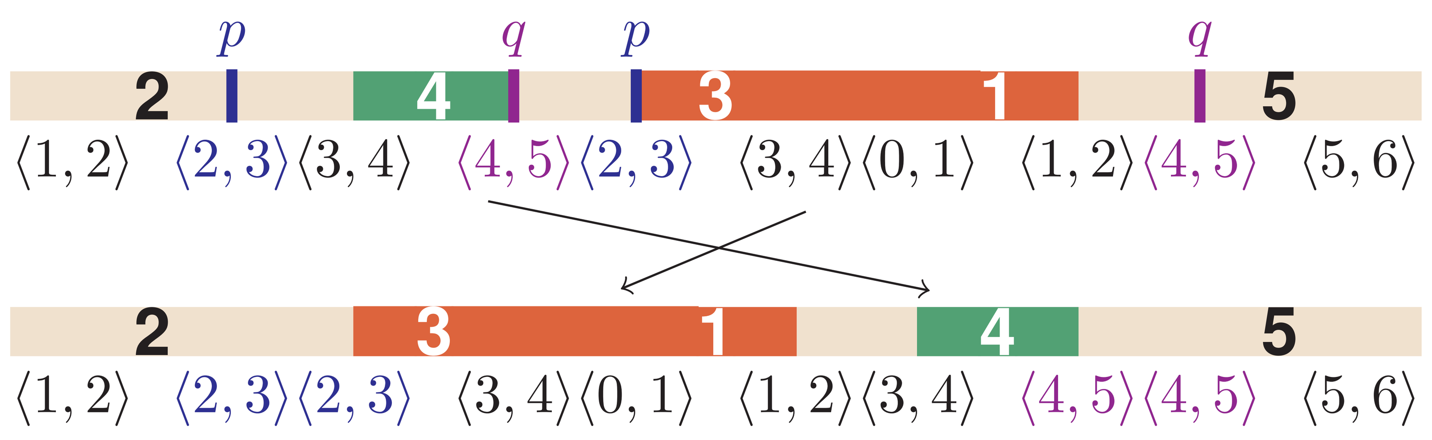

Figure 3: CDS on the permutation [2 4 3 1 5]

A signed (unsigned) permutation π is a **CDR (CDS) fixed point** when no valid CDR (CDS) moves exist. [1]

Reachability and Sortability

A signed permutation τ is **reachable** from a signed permutation π if valid applications of CDR or CDS to π produce τ. If only CDR is used, τ is **CDR reachable** from π. If $\tau = [1\ 2 \ldots n]$ is reachable from π, we say π is **mixed sortable**; and if $\tau = [-n \ldots -2\ -1]$ is reachable from π, π is **mixed reverse sortable**. If $\tau = [1\ 2 \ldots n]$ is CDR reachable from π, we say π is **CDR sortable**. When $\tau = [-n \ldots -2\ -1]$ is CDR reachable from π, π is **reverse CDR sortable**. Otherwise, π is **non-sortable**.

Theorem 1

For a CDR fixed point f and a signed permutation π, f is CDR reachable from π if, and only if, $f^{-1}\pi$ is CDR sortable.

Cycle Transformations

The **cycle transformation** of π is defined as $D_\pi = Z_\pi \circ W_n$, where $W_n = (n+1\ -(n+1)\ -n \ldots -1\ 0\ 1 \ldots\ n)$ and $Z_\pi = (-(n+1)\ (n+1)\ \pi_n \ldots \pi_1\ 0\ -\pi_1\ \ldots\ -\pi_n)$.

$$\pi = [2\quad -4\quad 6\quad 1\quad -3\quad 5]$$

$$D_\pi = (-7\ \ 7\ \ 5\ \ -3\ \ 1\ \ 6\ \ -4\ \ 2\ \ 0\ \ -2\ \ 4\ \ -6\ \ -1\ \ 3\ \ -5) \circ (7\ \ -7\ \ -6\ \ -5\ \ -4\ \ -3\ \ -2\ \ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$$

The **strategic stack** of π, $SS(\pi)$, is the set of entries following n when $\alpha_\pi = \beta_\pi$, and $\emptyset$ otherwise.

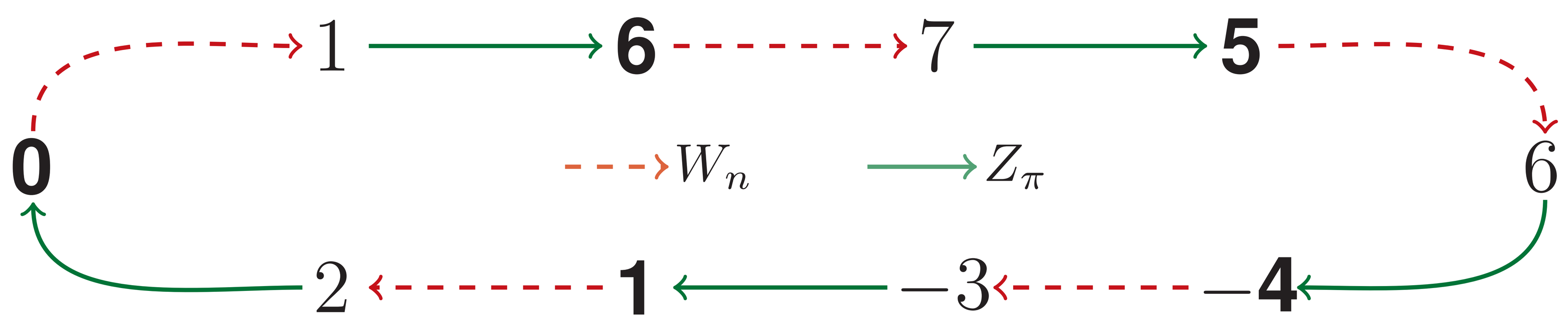

Figure 4: The first disjoint cycle of the above example

$$= (\mathbf{0\ 6\ \underline{5\ {-4}\ 1}})(-1\ -2\ \ 3\ -6\ -7)(2\ -5)(-3\ \ 4)$$

Experimental data suggest that the strategic stack corresponds to CDS fixed points of the permutation.

Theorem 2

For a given element i of a signed permutation π, doing a CDR move on the $\langle i, i+1\rangle$ pointer results in i and $-(i+1)$ being removed from their respective cycles in D_π.

D_π and Sortability

For $x \in \mathbb{Z}$, we define $d(x) = -(x+1)$ to be the **dual** of x.

Theorem 3

A cycle $\phi = (a_1 \ldots a_k)$ is contained in D_π if, and only if, the **dual cycle** of ϕ, $d(\phi) = (d(a_k) \ldots d(a_1))$, is contained in D_π.

Let α_π be the cycle in D_π containing 0 and let β_π be the cycle in D_π containing n.

Theorem 4

Let π be a signed permutation. Then,

(a) If $\alpha_\pi = \beta_\pi$, then π is non-sortable.
(b) If $\alpha_\pi = d(\beta_\pi)$, then π is mixed reverse sortable.
(c) If $\alpha_\pi \neq \beta_\pi$ and $\alpha_\pi \neq d(\beta_\pi)$, then π is mixed sortable.
(d) If there exists in D_π a cycle of nonnegative elements which appear in π (including zero), then π is not CDR sortable.

Future Work

- Extend Theorem 1 to general pairs of signed permutations.
- Prove the observed relationship between strategic stacks and CDS fixed points.
- Characterize CDR game outcomes on signed permutations.
- Prove the converse of Theorem 4 part (d) to get a full CDR sortability criterion.

References

[1] K.L.M. Adamyk et al., *Sorting Permutations: Games, Genomes and Cycles*, **arXiv:1410.2353**.
[2] D.M. Prescott, *Genome Gymnastics: Unique Modes of DNA Evolution and Processing in Ciliates*, **Nature Reviews Genetics**, 1:3 (2000), 191 - 198.

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