GEOMETRY, TOPOLOGY, AND COMPLEXITY OF VIRTUAL KNOTS

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1. Abstract

Knots, which are strings tangled in 3-space, are objects encountered in daily life. The mathematical theory of knots is highly sophisticated, incorporating many classical areas including topology, geometry, combinatorics and group theory. Currently, the study of knots is finding application in fields as diverse as biology, physics and computing.

When drawn on a piece of paper, a classical knot is a planar 4-regular graph. A virtual knot, from a graph theoretic point of view, is an arbitrary (not necessarily planar) 4-regular graph. Many questions which have been answered for classical knots are still unanswered for virtual knots. Virtual knots play key roles in long standing conjectures, such as Whitehead's asphericity conjecture. These REU projects are concerned with the topology, geometry and complexity of virtual knots. We will investigate generic properties of virtual knots with a large number of crossings and use computers for testing some of our conjectures.

2. Background

Much information regarding both classical and virtual knots can be found on the internet. The Wikipedia page on knot theory provides a good introduction to knots, as does the wiki The Knot Atlas. Rolfson's knot table contains all knots up to 10 crossings. Clicking on a knot in the table gives a list of properties and invariants of the knot. There is also a Wikipedia page that contains information on virtual knots. Jeremy Green maintains a table of virtual knots. More information on the geometry and topology of virtual knots can be found on my webpage

3. Detailed Project Descriptions

3.1. **Project on generic properties of virtual knots.** This project exposes the students to graph theory, topology and geometry in dimension 2, knot theory, and 3-manifolds.

Let P be a property that a virtual knot might have, such as alternating, aspherical, or hyperbolic. We say that virtual knots *generically have property* P, if among all virtual knots with at most n crossings, most will have property P when n is sufficiently large. We are interested in studying the geometry and topology of a virtual knot from a generic viewpoint. Given a virtual knot, that is a 4-regular graph in the plane with over and under crossing information at some vertices, it is not difficult to construct a surface of minimal genus on which the virtual knot embeds. We call such a surface a projection surface of the virtual knot. The virtual knot can now be viewed as the 1-skeleton of a tessellated surface.

A natural question one could explore is the nature of the projection surfaces of virtual knots. Specifically, among virtual knots with at most n crossings, how many have the 2-sphere as their projection surface, how many have the torus as their projection surface, and how many have a surface of genus higher than one as their projection surface? One can approach this question both computationally and combinatorially.

Once a projection surface for a virtual knot is known, it can be used to derive geometric properties of the virtual knot (see Harlander [3]). This can be used to address questions regarding the curvature of the spines of various virtual knot compliments. Specifically, among alternating virtual knots with at most n crossings, how many have a Wirtinger complex or a Dehn complex that is a non-positively curved square complex?

3.2. **Project on spherical pictures over virtual knots.** This project exposes the students to graph theory, 2-complexes, combinatorial homotopy theory, and knot theory.

The Whitehead conjecture, stated in 1941 by J. H. C. Whitehead, implies that every map from a 2-sphere into the Wirtinger complex of a long virtual knot (here *long* means that the ends of the knot are not tied together) can be deformed to a constant map. A map from a 2-sphere into the Wirtinger complex can be encoded by a spherical picture over the virtual knot. Such a spherical picture is a labeled link projection on a 2-sphere, such that every labeled crossing appears also as a labeled crossing in the virtual knot.

It was observed by Rosebrock [5], that long virtual knots admit reduced spherical pictures. This indicates that the Whitehead conjecture can not be disproven using exclusively combinatorial techniques. However, there remain many interesting open questions concerning the combinatorial homotopy theory of virtual knots. Recently Rosebrock [4] has been using computational methods to produce examples of reduced spherical picture over a long virtual prime knot, but these examples remain isolated. For this project we will attempt to develop methods for systematically constructing reduced spherical pictures over long virtual knots and investigate why no reduced spherical pictures for an alternating long virtual knot have been found.

References

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[2] J. Harlander, *On the topology and geometry of virtual knots*, in preparation.

[3] J. Harlander, *Hyperbolic alternating virtual link groups*, to appear in Groups, Geometry, and Dynamics, 2012

[4] J. Harlander, S. Rosebrock, *On primeness of labeled oriented trees*, to appear in Journal of Knot Theory and its Ramifications, 2012.

[5] S. Rosebrock, *A reduced spherical diagram into a ribbon–disk complement and related examples*, Editor: P. Latiolais, Lecture notes in mathematics 1440, Springer Verlag (1990), 175-185.