Big Ideas

- A **surface** in $\mathbb{R}^3$ is a 2-dimensional object in 3-space.
- Surfaces can be described using two variables. One way of doing this is to use a vector valued function (parameterization).
- Holding one parameter at a time constant while the other is allowed to vary gives rise to two families of **grid curves** (aka traces).
- Just as the line elements $ds$ and $dr$ are used to measure distance and direction of travel along a curve, **surface elements** $dS$ and $dS$ are used to measure area and the normal direction on a surface.

What We Are Doing Today . . .

So far, we have found ways to integrate over curves — scalar and vector line integrals.

Now, we want to be able to integrate over surfaces.

To do this, we will need to:

- Parameterize the surface — this gives us a way of writing our integral in terms of two variables (the parameters).
- Find a way to measure area on the surface — $dS$, called the **scalar surface element** — and combine that with a vector normal to the surface — $dS$, called the **vector surface element**.
Surfaces in $\mathbb{R}^3$

A surface in $\mathbb{R}^3$ is a 2-dimensional object sitting in 3-space.

Some examples of surfaces:

- Planes: $ax + by + cz = d$.
- Cylinders: $x^2 + y^2 = R^2$.
- Cones: $z = c\sqrt{x^2 + y^2}$.
- Spheres: $x^2 + y^2 + z^2 = R^2$.

Because surfaces are 2-dimensional, it makes sense that we should be able to express them in terms of two independent variables . . .

Some Surfaces are Graphs of Functions $z = f(x, y)$

For example:

- Planes: $z = D - Ax - By$.
- Cones: $z = c\sqrt{x^2 + y^2}$.
- Hemispheres: $z = \sqrt{R^2 - (x^2 + y^2)}$, or $z = -\sqrt{R^2 - (x^2 + y^2)}$. 

Review for this Topic:

- Vector-valued functions/parametrized curves (sec 13.1).
- Partial derivatives (sec 14.3).
- Cylindrical and spherical coordinates (sec 15.8, 15.9).
Some Surfaces are Not Graphs of Functions
\[ z = f(x, y) \]

For example:
- spheres:  \[ x^2 + y^2 + z^2 = R^2 \]
- cylinders:  \[ x^2 + y^2 = R^2, \quad a \leq z \leq b \]

Parameterizations of Surfaces

Surfaces can be parameterized using a vector function (called the parameterization):
\[ r(u, v) = x(u, v) \hat{i} + y(u, v) \hat{j} + z(u, v) \hat{k} \]

- The variables \( u \) and \( v \) are called parameters.
- \( x(u, v), y(u, v), z(u, v) \) are called coordinate (or component) functions.

Parameterizations of Curves vs. Surfaces

**Curve in \( \mathbb{R}^3 \):**
\[ r(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \]
- Tells you where you are on the curve with respect to a single parameter \( t \).
- One parameter \( t \), because curves are one-dimensional.

**Surface in \( \mathbb{R}^3 \):**
\[ r(u, v) = x(u, v) \hat{i} + y(u, v) \hat{j} + z(u, v) \hat{k} \]
- Tells you where you are on the surface with respect to two parameters \( u \) and \( v \).
- Two parameters \( u, v \), because surfaces are two-dimensional.
Example: Parameterization of Sphere

Equations:
\[ x^2 + y^2 + z^2 = 9 \text{ (Cartesian)} \quad \rho = 3 \text{ (spherical)} \]

Cartesian Coordinates:
upper hemisphere: \[ r_+(x, y) = x \hat{i} + y \hat{j} + \sqrt{9 - (x^2 + y^2)} \hat{k} \]
lower hemisphere: \[ r_-(x, y) = x \hat{i} + y \hat{j} - \sqrt{9 - (x^2 + y^2)} \hat{k} \]

Spherical Coordinates:
\[ r(\phi, \theta) = 3 \sin \phi \cos \theta \hat{i} + 3 \sin \phi \sin \theta \hat{j} + 3 \cos \phi \hat{k} \]

Grid Curves (Traces)

Suppose a surface \( S \) is parametrized by \( r(u, v) \).

The grid curves (or traces) of the parameterization are the two families of curves on \( S \) obtained by holding one parameter constant while allowing the other to vary.

Example: Grid Curves (Traces)

Grid curves of the \( xy \)-coordinate plane, parametrized using:
- Cartesian: \( r(x, y) = x \hat{i} + y \hat{j} + 0 \hat{k} \).
- Polar: \( r(r, \theta) = r \cos \theta \hat{i} + r \sin \theta \hat{j} + 0 \hat{k} \).

Grid curves of the cone \( z = \sqrt{x^2 + y^2} \), parametrized using:
- Cartesian: \( r(x, y) = x \hat{i} + y \hat{j} + \sqrt{x^2 + y^2} \hat{k} \).
- Polar: \( r(r, \theta) = r \cos \theta \hat{i} + r \sin \theta \hat{j} + r \hat{k} \).
Surface Elements

- The **scalar surface element** $dS$ is the “area” of an infinitesimal rectangle on a surface $S$.
- The **vector surface element** $d\mathbf{S}$ is a vector normal to a surface $S$, with magnitude $dS = |d\mathbf{S}|$.
- The scalar surface element $dS$ gives a measure of how much the area of an infinitesimal rectangle $dA = du \, dv$ changes under a parameterization $r(u, v)$.

Computing Surface Elements Given a Parametrization

$$r(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$$

The differentials $dr_u = r_u(u, v) \, du$ and $dr_v = r_v(u, v) \, dv$ are tangent to the surface, lying along the grid curves, and form the sides of a parallelogram on the surface. By the properties of the cross product, $dr_u \times dr_v$ is normal to the surface, and $|dr_u \times dr_v|$ is the area of the parallelogram.

$$dS = dr_u \times dr_v = \left( r_u(u, v) \times r_v(u, v) \right) \, du \, dv$$

$$dS = |dS| = |dr_u \times dr_v| = \left| r_u(u, v) \times r_v(u, v) \right| \, du \, dv$$