ARITHMAGONS AND GEOMETRICALLY INVARIANT MULTIPLICATIVE INTEGER PARTITIONS

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ABSTRACT. In this article, we introduce a formal definition for integral arithmagons. Informally, an integral arithmagon is a polygonal figure with integer labeled vertices and edges in which, under a binary operation, adjacent vertices equal the included edge. By considering the group of automorphisms for the associated graph, we count the number of integral arithmagons whose exterior sum or product equals a fixed number.

1. Introduction

Arithmagons are numerical problems in which solvers look for ways to label a polygonal graph so that numbers on adjacent vertices combine under a binary operation to equal numbers on the edges as shown in Figure 1.

As a flexible context for generating problems of varying difficulty, arithmagons present nice opportunities for school-aged students to build algebraic reasoning skills while exploring properties of numbers and operations [6, 7]. The main question we explore in this article relates to how many such arithmagons can be constructed if the combined binary operation on the values on the edges must equal a fixed number $N$. We begin by considering multiplicative arithmagons. We then use this result to count additive arithmagons using a well-known relation between partitions of $N$ and factorizations of $q^N$ for any prime $q$. We obtain a well-defined count for
both operations whenever all nonidentity vertex values are distinct. We also provide a Mathematica® function for obtaining these counts.

The paper is organized as follows. In Section 2, we treat arithmagons as a type of multigraph satisfying a system of equations. Conditions for the existence of a solution are provided and general solutions are given explicitly. In Section 3, we use the automorphism groups of the associated multigraphs to determine the number of distinct arithmagons obtainable from a given set of vertex entries. This is used to address the following formal problem: Let \( m(v_i, v_{i+1}) \) denote the value on the edge that connects the vertices \( v_i \) and \( v_{i+1} \) of an arithmagon. Our goal is to find the number of distinct arithmagons with \( n \) vertices (up to rotations and reflections) that satisfy

\[
m(v_1, v_2) \ast m(v_2, v_3) \ast \cdots \ast m(v_{n-1}, v_n) \ast m(v_n, v_1) = N \tag{1.1}
\]

for a fixed \( N \in \mathbb{N} \), where \( \ast \) is either addition or multiplication of integers.

2. Formal Definition of an Arithmagon

In this section, we offer a formal definition of arithmagons. Loosely speaking, an arithmagon will be defined as the orbit of an undirected labeled multigraph.

To make this precise, let \( \Gamma = (V,E) \) be an undirected, labeled, polygonal multigraph, with vertex set \( V \) and multiset of edges \( E \). Since \( \Gamma \) is polygonal, we can order the vertices as \( V = \{v_1,\ldots,v_n\} \) and record the multiplicity of the edges as \( E = (A,m) \), where \( A = \{(v_i,v_{i+1}) \mid 0 \leq i \leq n - 1\} \cup \{(v_n,v_1)\} \) and \( m : A \to \mathbb{Z}^>0 \) is the multiplicity function.

It is well-known that if a permutation \( \sigma \) is an automorphism of the graph \( \Gamma \), then \((\sigma(u),\sigma(v)) \in E\) if and only if \((u,v) \in E\) and that the set of all of these maps under composition forms the automorphism group \( G = \text{Aut} \Gamma \), [1]. Since we restrict our attention to polygonal graphs, we have \( G = D_{2n} \), where \( n \) is the number of vertices in \( V \).

Let \( p : V \to \mathbb{Z}^>0 \) be a map such that

\[
p(i) \ast p(i + 1) = m(v_i, v_{i+1}) \tag{2.1}
\]

\[
p(n) \ast p(1) = m(v_n, v_1), \tag{2.2}
\]

where \( \ast \) is either addition or multiplication of integers. To simplify notation, we will define \( m(v_n, v_{n+1}) := m(v_n, v_1) \).

An important remark is that these conditions are equivariant under the action of \( G \). This compatibility allows us to define an arithmagon in the following way.

Definition 1. A **positive integral multiplicative (or additive) arithmagon** is the orbit of an undirected, labeled, polygonal multigraph \( \Gamma = (V,A,m) \) under the action of \( G = \text{Aut} \Gamma \) and a map \( p : V \to \mathbb{Z}^>0 \) such that Equations (2.1) and (2.2) are satisfied by the multiplicative (or additive) operation \( \ast \).

In this paper, we restrict ourselves to positive arithmagons. However, we could define more general integral arithmagons by considering directed multigraphs, where all edges between any pair of vertices have the same direction. We would say that the edge \((v_i, v_j)\) is positively oriented if \( i < j \) and negatively oriented if \( i > j \). This would allow us to identify positive and negative edges on the arithmagon. In this case, the map \( p \) would be allowed to take values in \( \mathbb{Z} \).

A few questions are in order about the definition. First, under what conditions is an undirected, labeled, polygonal multigraph an arithmagon? The answer of this
question is the same as asking whether an arithmagon is solvable for a known set of edge labels. The additive case is simple, and it is explained in the following proposition.

**Proposition 1.** Let $\Gamma = (V = \{v_1, ..., v_n\}, A, m)$ be undirected, labeled, polygonal multigraph. A map $p : V \to \mathbb{Z}$ such that Equations (2.1) and (2.2) are satisfied with addition of integers as $\ast$, exists if and only if

$$\begin{cases} \sum_{i=1}^{n} m(v_i, v_{i+1}) = 0 \pmod{2} & \text{if } n \text{ is odd} \\ \sum_{\text{even } i} m(v_i, v_{i+1}) = \sum_{\text{odd } j} m(v_j, v_{j+1}) & \text{if } n \text{ is even.} \end{cases}$$

If these conditions are satisfied, the map $p$ is defined in the following way

$$p(i) = \begin{cases} (-1)^i p(n) + \sum_{j=1}^{n-1} (-1)^{i+j} m(v_j, v_{j+1}) & \text{if } n \text{ is even} \\ \frac{1}{2} \sum_{j=1}^{n} (-1)^{i+j} m(v_j, v_{j+1}) + \frac{1}{2} \sum_{j=1}^{n-1} (-1)^{i+j-1} m(v_j, v_{j+1}) & \text{if } n \text{ is odd,} \end{cases}$$

where $p(n)$ is an arbitrary parameter when $n$ is even and determined by the previous equation if $n$ is odd.

**Proof.** Conditions (2.1) and (2.2) give a system of equations. Standard linear algebra shows the system is always consistent and has a unique solution when $n$ is odd. The condition $\sum_{i=1}^{n} m(v_i, v_{i+1}) = 0 \pmod{2}$ guarantees that the unique solution will be in $\mathbb{Z}$. When $n$ is even, the system is consistent if and only if $\sum_{\text{even } i} m(v_i, v_{i+1}) = \sum_{\text{odd } j} m(v_j, v_{j+1})$ is satisfied. In this case, the solution to the system will always be integral.

We have shown necessary and sufficient conditions for the existence of a map $p$ from $V$ to $\mathbb{Z}$. Thus it is worth noting that, although the associated arithmagon will have positive integer entries on the edges, the entries on the vertices may be positive or negative integers.

The analogous results for the multiplicative arithmagons can be obtained similarly using logarithms.

**Proposition 2.** Let $\Gamma = (V = \{v_1, ..., v_n\}, A, m)$ be an undirected, labeled, polygonal multigraph. A map $p : V \to \mathbb{Q}_{>0}$ such that Equations (2.1) and (2.2) are satisfied with multiplication as the operation $\ast$ exists if and only if

$$\begin{cases} \sqrt[n]{\prod_{i=1}^{n} m(v_i, v_{i+1})} \in \mathbb{Z} & \text{if } n \text{ is odd} \\ \prod_{\text{even } i} m(v_i, v_{i+1}) = \prod_{\text{odd } j} m(v_j, v_{j+1}) & \text{if } n \text{ is even.} \end{cases}$$

Moreover, if $n$ is odd, the map $p$ is uniquely defined.

3. **Counting Problems**

With the formal definition of arithmagons at hand, we can start counting the number of arithmagons for which the entries on the edges multiply or add to a fixed natural number. More precisely, we seek to count how many arithmagons satisfy

$$m(v_1, v_2) \ast m(v_2, v_3) \ast \cdots \ast m(v_{n-1}, v_n) \ast m(v_n, v_1) = N \quad (3.1)$$

for a fixed $N \in \mathbb{N}$.

By Equations (2.1) and (2.2), condition (3.1) becomes:

$$(p(1) \ast p(1)) \ast \cdots \ast (p(n) \ast p(n)) = N. \quad (3.2)$$

If $\ast$ is addition, this implies $2 \sum_i p(i) = N$. Thus, $N$ is necessarily an even number. If $\ast$ is multiplication, this implies $\prod_i p(i)^2 = N$ must be a perfect square.
3.1. Symmetry Considerations. We will look at the counting problem for the additive and multiplicative cases separately. However, in both cases, we will take advantage of some symmetry considerations. To this end, we will consider the action of $S_n$ on the arithmagons in more detail.

The standard action of $S_n$ on a graph maps an edge $(v_i, v_j)$ to $(\sigma(v_i), \sigma(v_j))$. If $\sigma \in \text{Aut } \Gamma$, then $(\sigma(v_i), \sigma(v_j))$ is also an edge of $\Gamma$. This is not true for any other permutation in $S_n$. In particular, the standard action of $S_n/G$ does not preserve the structure of an arithmagon. Therefore, we define the action of $S_n$ as follows,

$$\sigma \cdot (V, A, m) = (\sigma \cdot V, A, \sigma \cdot m),$$

(3.3)

where $\sigma \cdot m : A \to \mathbb{Z}_{>0}$ is the unique map that satisfies conditions (2.1) and (2.2) with the new order of the vertices. We obtain (1) as an immediate consequence of this definition.

**Lemma 1.** The action defined in (3.3) restricts to the standard action of $G \subset S_n$.

While the action definition fixes problems with the standard action of $S_n$ on arithmagons, it does not affect the action of the automorphism group $G$. Moreover, since $S_n$ acts transitively on $V = \{v_1, \ldots, v_n\}$ and $V$ determines $m$ uniquely, $S_n$ acts transitively on the space of arithmagons with equal entries.

In order to count distinct arithmagons satisfying (3.2), we need to fix a set of entries on the vertices $S = \{p(i) \mid 1 \leq i \leq n\}$ which satisfy the condition and then count how many configurations yield a different arithmagon. To do this, we will calculate the stabilizer of the arithmagon with entries in $S$. This calculation depends on the number of indices for which $p(i) = p(j)$. We consider the partition, $I$, on the set of indices induced by the equivalence relation $i \sim j$ if and only if $p(i) = p(j)$. Toward that end, form the list $I = \{[a_1], \ldots, [a_l], [a_{l+1}], \ldots, [a_k]\}$ where $||a_i|| = 1$ for all $1 \leq i \leq l$ and $||a_i|| > 1$ for all $l + 1 \leq i \leq k$.

**Lemma 2.** Let $\Gamma = (V, A, m)$ be an arithmagon with vertex entries in $S = \{p(i) \mid 1 \leq i \leq n\}$. Define $I$ and its classes as in the previous paragraph. Let $G_{\Gamma}$ be the stabilizer of $\Gamma$. Then, the number of arithmagons with vertex entries in $S$ is given by

$$s(\Gamma) := |S_n \cdot \Gamma| = \frac{n!}{|G_{\Gamma}|}.$$

and

$$|G_{\Gamma}| = \begin{cases} 2n \prod_{i=l+1}^{k} ||a_i||! & \text{if } l \geq 3 \\ 2 \cdot (n-1)! & \text{if } l = 2 \text{ and } k \leq 3 \\ n! & \text{if } l = 1 \text{ and } k \leq 2 \end{cases}$$

when $n$ is even, and

$$|G_{\Gamma}| = \begin{cases} 2n \prod_{i=l+1}^{k} ||a_i||! & \text{if } l \geq 2 \\ n! & \text{if } l = 1 \text{ and } k \leq 2 \end{cases}$$

when $n$ is odd.

**Proof.** Since the action of $S_n$ is transitive, the first statement follows from the Orbit-Stabilizer Theorem. Suppose that $l \geq 3$. Since the arithmagon is invariant under the permutation of the edges that are labeled with $a_i$ for $i \geq l + 1$, a copy of $S_{||a_i||}$ can be embedded in the stabilizer of $\Gamma$ for each $l + 1 \leq i \leq k$. Moreover, the actions of these symmetric groups commute with each other. Now, each element
in $D_{2n}$ leaves either 0 or 2 points fixed on the arithmagon if $n$ is even and 0 or 1 if $n$ is odd, but $l \geq 3$ guarantees that any permutation in any of the $S_{[a_i]}$ will fix at least 3 points, therefore these permutations do not generate any element of $D_{2n}$ and vice-versa. This shows that when $l \geq 3$, $|G| = 2n \prod_{i=1}^{k} |[a_i]|$. The previous argument still holds true when $n$ is odd and $l = 2$ because in this case reflections have a single fixed point.

If $n$ is even, $l = 2$, and $k = 3$, we have one number that appears $n - 2$ times and two different numbers appear just once. Therefore, all the possible arithmagons consisting of two elements that appear just once are separated by $0, 1, \ldots, \text{or } n/2 - 1$ many copies of the same number. This shows that the stabilizer has order $2(n-1)!$.

Independently of the parity of $n$, if $l = 1$ and $k \leq 2$, then either, there is one number that appears $n - 1$ times and another that appears just one time, or there is a number that appears $n$ times. In both cases, the resulting arithmagon is stabilized by $S_n$. □

3.2. Counting Problem for Multiplicative Arithmagons. In Lemma 2, we determined the number of different arithmagons that can be obtained by permuting the vertices given a fixed set of vertex entries. So, if we want to find the number of arithmagons that satisfy condition (3.2), then we need to find the number of entries in the vertices that would satisfy this condition and multiply it by the number of arithmagons that can be obtained through permutations. In the case of multiplicative arithmagons, this would mean counting the number of factorizations of $\sqrt{N}$ into at most $n$ factors. This is an open problem in number theory. Applications of recursive methods using Mathematica® can be found in [5], and a survey on recent advances can be found in [4].

We start with the simpler, but interesting case when $\sqrt{N} = q_1 \cdots q_m$ is a perfect square.

**Proposition 3.** If $n$ is odd, then the number of integral, positive multiplicative $n$-arithmagons with edge values multiplying to $N = q_1^2 \cdots q_m^2$ (i.e. that satisfy (3.2)) is given by:

$$1 + \frac{1}{2} \sum_{l=2}^{n} \frac{(n-1)!}{(n-l)!} S(m, l),$$

where $S(m, l)$ represents the Stirling number of the second kind. If $n$ is even, the number of such arithmagons is equal to

$$1 + \frac{n}{2} S(m, 2) + \frac{1}{2} \sum_{l=3}^{n} \frac{(n-1)!}{(n-l)!} S(m, l).$$

**Proof.** It is well-known that the number of factorizations of a square free number $\sqrt{N} = q_1 \cdots q_m$ in $l$ factors is given by the Stirling number of the second kind, $S(m, l)$ (see Chapter 6 of [2]). Since $\sqrt{N} = q_1 \cdots q_m$, either $p(i) \neq p(j)$ or $p(i) = p(j) = 1$ when $i \neq j$. Then $l$ is the number of entries that are not equal to 1 and $n-l$ is equal to the number of vertices that must get a 1 assigned to them. By Lemma 2, the stabilizer of the arithmagon is of order either $n!$, $2(n-1)!$, or $2n(n-1)!$.

Now, the proposition follows from multiplying each $S(m, l)$ by its corresponding order and adding each of these terms. □
The assumption that √N is a square-free number is rather stringent. We can solve a slightly different counting problem by using the recursive algorithm presented in [5], which is based on Hughes-Shallit’s reasoning in [3]. In particular, we can count the number of multiplicative arithmagons with n vertices whose vertex entries multiply to (any perfect square) N and all non-identity vertex labels are different.

Theorem 1. Let \( P_d(l,N) \) denote the number of multiplicative partitions of N in l distinct factors. The number of multiplicative arithmagons with n vertices satisfying (3.2) for any perfect square N and such that the only entry that can appear more than once is 1 is equal to:

\[
\sum_{l=1}^{n} \left\lceil \frac{(n-1)!}{2(n-l)!!} \right\rceil P_d(l,\sqrt{N}).
\]

Proof. The proof mimics the proof of Proposition 3 and thus, is omitted. □

Remark 1. Notice that Proposition 3 is a special case of Theorem 1 since \( P_d(l,\sqrt{N}) = S(m,l) \) when \( \sqrt{N} = q_1 \cdots q_m \), and because the latter condition forces the non-unit entries on the vertices to be distinct.

An interesting result occurs when \( N = q^{2r} \) for some prime q. It is well-known that the number of factorizations of \( \sqrt{N} \) in l factors is the same as the number of partitions of r in exactly l parts (c.f. [4]). This is because a partition \( \{r_1, \ldots, r_l\} \) of r gives an unordered factorization \( q^{r_1} \cdots q^{r_2} \) of \( \sqrt{N} \) and vice-versa. This induces a natural equivalence between multiplicative arithmagons whose exterior product is of the form \( q^{2r} \) and the additive arithmagons whose exterior sum equals 2r. Two equivalent multiplicative and additive pentagonal arithmagons are shown in Figure 2. We record this in the following proposition.

![Figure 2. Equivalence between a multiplicative and an additive pentagonal arithmagon.](image)

Proposition 4. The number of multiplicative arithmagons with exterior multiplication equal to \( N = q^{2r} \) is the number of additive arithmagons with exterior sum equal to 2r.

3.3. Counting Problem for Additive Arithmagons. As a result of Proposition 4 and Theorem 1, we can count the number of additive arithmagons for which all non-zero entries are distinct.
Corollary 1. The number of additive arithmagons that satisfy (3.2) for \( N \in 2\mathbb{N} \) and whose all non-zero entries are distinct is equal to

\[
\sum_{l=1}^{n} \left\lceil \frac{(n-1)!}{2(n-l)!} \right\rceil P_d(l, q^{N/2})
\]

for any prime \( q \).

Figure 3. Positive multiplicative pentagonal arithmagons with an exterior product \( N = 900 \).

4. Example and Conclusions

To illustrate the results, consider the problem of counting the number of integral, positive multiplicative pentagonal arithmagons \( (n = 5) \) with an exterior product
$N = 900 = 2^23^25^2$. Since $\sqrt{N} = 30$ is a square-free number, we apply Proposition 3 to obtain that the number of such arithmagons is

$$1 + \sum_{l=2}^{5} \frac{4!}{2(5-l)!} S(3, l) = 13.$$  

These 13 pentagonal arithmagons are shown in Figure 3. To support the computations, we provide a Mathematica® function we found useful for calculating the number of multiplicative arithmagons with distinct non-unit entries that have an exterior product equal to $N$. Applying Corollary 1, the function can be used to calculate the number of additive arithmagons whose exterior sum equals $N$ by evaluating $s(q^n, n)$ for some prime $q$.

The code from [5] that we will use for our calculation is the following:

```mathematica
DistinctUnorderedFactorizations[m_,1] = {{}};
DistinctUnorderedFactorizations[1,n_] = {{}};
DistinctUnorderedFactorizations[0,n_] = {{}};
DistinctUnorderedFactorizations[m_,n_/;PrimeQ[n]] := If[m<n,{n}]
DistinctUnorderedFactorizations[m_,n_]:=DistinctUnorderedFactorizations[m,n]=Flatten[Function[d,Prepend[#,d]/@DistinctUnorderedFactorizations[d-1,n/d]]/@Rest[Select[Divisors[n],#<=m&]],1]
DistinctUnorderedFactorizations[n_]:=DistinctUnorderedFactorizations[n,n]
```

Using this code we can now write the code for the function that calculates the number of arithmagons:

```mathematica
s[N_,n_]:=Plus @@ (#*Ceiling[(n-1)!/(2(n-Range[Length[#]])!)]&[Length /@ Split[Sort[DistinctUnorderedFactorizations[Sqrt[N]],Length[#1]<Length[#2]&],Length[#1]==Length[#2]&]]
```

4.1. Conclusions. We started this article with a formal definition of an arithmagon that is compatible with the intuitive definition and that offers the additional structure of graphs. This introduced consideration of symmetries and the group of automorphisms of the graph. Using these symmetries and recursive methods, we solved counting problems pertaining to the number of multiplicative and additive arithmagons with no repeated non-unit entries. In the case of multiplicative arithmagons, this included all possible arithmagons when $N$ is the perfect square of a square-free number. We illustrated one of the main results with the help of a Mathematica® function based on [5].

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