Partial fraction decomposition (PFD) review.

Consider two polynomials:
\[ Q(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m \]
\[ P(x) = b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n \]

with \( m < n \).

- **Goal:** we want to write the rational function \( \frac{Q(x)}{P(x)} \) as the sum of rational functions having only powers of linear and quadratic polynomials in the denominators (a rational function is the quotient of two polynomials; a linear polynomial is a first-degree polynomial; a quadratic polynomial is a second-degree polynomial).

- **Reason:** in general, it is easier to integrate a rational function whose denominator is the power of a first- or second-degree polynomial than one whose denominator is of higher order. (There are occasional exceptions: for example, if the numerator is the derivative of the denominator, \( u \)-substitution can be used without resorting to PFD).

- **We can do it because:** (Theorem) it is possible to factor any polynomial as the product of powers of linear and quadratic terms.

**Method of partial fraction decomposition:**

1. Factor \( P(x) \) into a product of powers of linear and quadratic terms.

2. Each power of a linear term \( (x - r)^k \) appearing in the factorization of \( P(x) \) contributes the sum of partial fractions:
\[ \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \ldots + \frac{A_k}{(x - r)^k} \]
where \( A_1, \ldots, A_k \) are constants.

3. Each power of a quadratic term \( (x^2 + r_1 x + r_2)^k \) appearing in the factorization of \( P(x) \) contributes the sum of partial fractions:
\[ \frac{B_1 x + C_1}{x^2 + r_1 x + r_2} + \frac{B_2 x + C_2}{(x^2 + r_1 x + r_2)^2} + \ldots + \frac{B_k x + C_k}{(x^2 + r_1 x + r_2)^k} \]
where \( B_1, \ldots, B_k \) and \( C_1, \ldots, C_k \) are constants.
4. Add all resulting partial fractions. Put them over a common denominator (which will be \(P(x)\)). Write the numerator as a polynomial \(D(x)\). The coefficients of this polynomial will be linear combinations in the variables \(A_i, B_i\) and \(C_i\).

5. Equate the polynomials \(Q(x)\) and \(D(x)\). Since two polynomials of the same degree are equal if and only if the coefficients agree, this results in a system of linear equations in the unknowns \(A_i, B_i\) and \(C_i\). Note that some of the coefficients will likely be zero.

**Example 1** Consider the rational function:

\[
\frac{x^2 + 1}{x^3 - 3x - 2}
\]

Since \(x^3 - 3x - 2 = (x - 2)(x + 1)^2\), we get the partial fraction decomposition:

\[
\frac{x^2 + 1}{x^3 - 3x - 2} = \frac{A_1}{x - 2} + \frac{A_2}{x + 1} + \frac{A_3}{(x + 1)^2}
\]

\[
= \frac{A_1(x + 1)^2}{(x - 2)(x + 1)^2} + \frac{A_2(x + 1)(x - 2)}{(x + 1)(x + 1)(x - 2)} + \frac{A_3(x - 2)}{(x + 1)^2(x - 2)}
\]

\[
= \frac{A_1(x^2 + 2x + 1)}{x^3 - 3x - 2} + \frac{A_2(x^2 - x - 2)}{x^3 - 3x - 2} + \frac{A_3(x - 2)}{x^3 - 3x - 2}
\]

\[
= \left(A_1 + A_2\right) \frac{x^2}{x^3 - 3x - 2} + \left(2A_1 - A_2 + A_3\right) \frac{x}{x^3 - 3x - 2} + \left(A_1 - 2A_2 - 2A_3\right) \frac{-2}{x^3 - 3x - 2}
\]

Equating coefficients in the numerators gives the system of linear equations:

\[
A_1 + A_2 = 1
\]
\[
2A_1 - A_2 + A_3 = 0
\]
\[
A_1 - 2A_2 - 2A_3 = 1
\]

Solving this system, we have: \(A_1 = 5/9\), \(A_2 = 4/9\) and \(A_3 = -2/3\), so:

\[
\frac{x^2 + 1}{x^3 - 3x - 2} = \frac{5/9}{x - 2} + \frac{4/9}{x + 1} - \frac{2/3}{(x + 1)^2}
\]
Example 2 Consider the rational function:

\[
\frac{x^4 + 1}{x^3 - x^2 + x - 1}
\]

Observe that the order of the polynomial in the numerator is greater than that of the polynomial in the denominator. To remedy this situation, divide \(x^4 + 1\) by \(x^3 - x^2 + x - 1\):

\[
x^3 - x^2 + x - 1 \quad \overline{\quad x^4 + 0x^3 + 0x^2 + 0x + 1}
\]

\[
\qquad - x^4 + x^3 - x^2 + x
\]

\[
\qquad \boxed{x^3 - x^2 + x + 1}
\]

\[
\qquad - x^3 + x^2 - x + 1
\]

\[
\frac{2}{2}
\]

So:

\[
\frac{x^4 + 1}{x^3 - x^2 + x - 1} = x + 1 + \frac{2}{x^3 - x^2 + x - 1}
\]

Factoring the denominator yields: \(x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1)\), which leads to the partial fraction decomposition:

\[
\frac{2}{x^3 - x^2 + x - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}
\]

\[
= \frac{A(x^2 + 1)}{(x - 1)(x^2 + 1)} + \frac{(Bx + C)(x - 1)}{(x^2 + 1)(x - 1)}
\]

\[
= \frac{(A + B) x^2 + (-B + C) x + (A - C)}{x^3 - x^2 + x - 1}
\]

which gives the system of equations:

\[
A + B = 0
\]

\[
-B + C = 0
\]

\[
A - C = 2
\]

Solving this system, we see that \(A = 1\), \(B = -1\) and \(C = -1\), so:
\[
\frac{x^4 + 1}{x^3 - x^2 + x - 1} = x + 1 + \frac{2}{x^3 - x^2 + x - 1} = x + 1 + \frac{1}{x - 1} - \frac{x + 1}{x^2 + 1}
\]