Finding potential functions of conservative vector fields.

Recall the example from class (Monday 14 November):

$$\vec{F} = (e^x \cos y + yz) \hat{i} + (xz - e^x \sin y) \hat{j} + (xy + z) \hat{k}.$$ 

(In class, I called this field $\vec{G}$ in class; in these notes, I’m re-naming it $\vec{F}.$)

We computed the curl of $\vec{F}$ and found $\text{curl} \vec{F} = \vec{0}$, so we know $\vec{F}$ is conservative. The goal now is to find a potential function $f$ of $\vec{F}$ — that is, the scalar-valued function $f$ so that $\vec{F} = \nabla f$.

Since $\vec{F}$ is the gradient field of its potential function $f$, the following relations hold between the partial derivatives of $f$ and the components of $\vec{F}$:

$$\frac{\partial f}{\partial x} = e^x \cos y + yz$$

$$\frac{\partial f}{\partial y} = xz - e^x \sin y$$

$$\frac{\partial f}{\partial z} = xy + z.$$

The method presented in class for determining a potential function $f$ for the conservative field $\vec{F}$ consists of integrating each component of $\vec{F}$ (with respect to the appropriate variable) and comparing the resulting antiderivatives. First, integrate:

$$f(x, y, z) = \int e^x \cos y + yz \, dx = e^x \cos y + xyz + h_1(y, z)$$

$$f(x, y, z) = \int xz - e^x \sin y \, dy = xyz + e^x \cos y + h_2(x, z)$$

$$f(x, y, z) = \int xy + z \, dz = xyz + z^2/2 + h_3(x, y).$$
The purpose of the functions $h_i, \; i = 1, 2, 3$ is to account for any information lost in taking the partial derivatives — similar to the constant of integration for the antiderivative of a function of a single variable.

Now, compare the three antiderivatives. First, identify any terms common to all three:

$$f(x, y, z) = \int e^x \cos y + yz \, dx = e^x \cos y + xyz + h_1(y, z)$$

$$f(x, y, z) = \int xz - e^x \sin y \, dy = xyz + e^x \cos y + h_2(x, z)$$

$$f(x, y, z) = \int xy + z \, dz = xyz + z^2/2 + h_3(x, y).$$

The term $xyz$ is shared by all three, so:

$$f(x, y, z) = xyz + (\text{terms involving only one or two of the three variables}).$$

Now, find any terms shared by exactly two of the three antiderivatives:

$$f(x, y, z) = \int e^x \cos y + yz \, dx = e^x \cos y + xyz + h_1(y, z)$$

$$f(x, y, z) = \int xz - e^x \sin y \, dy = xyz + e^x \cos y + h_2(x, z)$$

$$f(x, y, z) = \int xy + z \, dz = xyz + z^2/2 + h_3(x, y).$$

The only term shared by two of the three antiderivatives is $e^x \cos y$, so:

$$f(x, y, z) = xyz + e^x \cos y + (\text{terms involving only one the three variables}).$$

Finally, identify the terms that appear in exactly one of the three antiderivatives:
\[ f(x, y, z) = \int e^x \cos y + yz \, dx = e^x \cos y + xyz + h_1(y, z) \]

\[ f(x, y, z) = \int xz - e^x \sin y \, dy = xyz + e^x \cos y + h_2(x, z) \]

\[ f(x, y, z) = \int xy + z \, dz = xyz + \frac{z^2}{2} + h_3(x, y). \]

The only term showing up in one of the antiderivatives is \( z^2/2 \). We now have a complete description of a potential function \( f \):

\[ f(x, y, z) = xyz + e^x \cos y + \frac{z^2}{2}. \]

Note that we never used the unknown functions \( h_1, h_2 \) and \( h_3 \) directly. They primarily acted as place-holders to remind us that, looked at individually, the components of the field \( \vec{F} \) do not tell us everything we need to know about a potential \( f \) (since partial derivatives can cause information about a potential function to be lost). But, if you are curious, they are given explicitly by:

\[
\begin{align*}
h_1(y, z) &= \frac{z^2}{2} \\
h_2(x, z) &= \frac{z^2}{2} \\
h_3(x, y) &= e^x \cos y.
\end{align*}
\]

This method can be adapted to finding potentials of two-dimensional fields in the natural way.