MIDTERM 3 REVIEW

The third midterm will be held in class (10:40–11:30pm) on Friday 4 November. You will be allowed one half of one side of an 8.5" × 11" sheet of paper (total surface area: 8.5" × 5.5") for handwritten notes, the table of basic derivatives and integrals posted on the course website, and a scientific calculator. No other notes or technology will be allowed for this exam.

This exam will be over material covered in class since the previous exam, corresponding to chapter 14 and section 15.1 in the text. Suggested review problems are included in this review. You should also review homework assignments and quizzes.

INTEGRATING FUNCTIONS OF TWO AND THREE VARIABLES

Key Ideas:

- Motivation: suppose \( f(x, y) \) is a non-negative continuous function defined on some region \( R \) in the \( xy \)-plane and you want to find the volume between \( R \) and the surface \( z = f(x, y) \). This volume can be approximated by partitioning \( R \) into rectangles and for each rectangle, choosing a point in the rectangle at which to evaluate \( f \). The rectangles are the bases of boxes, and the function evaluated at a point in a rectangle gives the height of the box for which the rectangle is the base. Adding up all of these boxes approximates the volume. The smaller the areas of the rectangles, the better the approximation. The double integral is the limit of these sums of boxes as the side-lengths of the boxes go to zero. One can also think of the double integral as adding up infinitely many boxes whose bases are infinitesimally small rectangles of area \( dA \). If \( f(x, y) = 1 \), the double integral is simply adding up all the infinitesimal rectangles to get the area of the region \( R \).
• The area element \( dA \) determines how area is measured in a double integral. The volume element \( dV \) determines how volume is measured in a triple integral.

• If the integrand is continuous, evaluation of multiple integrals is accomplished by evaluating a series of single integrals\(^1\) (Fubini’s theorem). If the boundaries allow, the order of integration may be changed without changing the value of the resulting integral. Being able to visualize the region of integration is crucial in determining limits of integration. You are expected to be able to sketch the following curves and surfaces:
  
  – curves: graphs of polynomials up to degree three (linear, quadratic, cubic); graphs of the square root and absolute value functions; hyperbolic, exponential, logarithmic, square root, sine, cosine, tangent and arctangent functions; equations of circles and ellipses.
  
  – surfaces: planes, cones, paraboloids, cylinders, hemispheres and spheres; surfaces generated by a function \( u = f(v) \) where \( f \) is one of the functions above (for example, the surface generated by \( z = y^2 \)).

• If a region or integrand possesses strong circular or spherical symmetry, converting to polar/cylindrical or spherical coordinates can greatly simplify integration. More generally, one can try to express the cartesian coordinates as functions of parameters; for example, in two dimensions, \( x = x(u, v), \ y = y(u, v) \). Integration can then carried out over a region in the \( uv \)-plane corresponding the the original region of integration in the \( xy \)-plane. The area element is determined by multiplying the the differentials \( du, dv \) of the parameters by the absolute value of the Jacobian, which is the determinant of the matrix whose columns are the partial derivatives of the coordinate functions. There is a natural generalization to changing coordinates for integrals over three-dimensional regions. (Compare this to finding \( du \) when carrying out a \( u \)-substitution in a single-variable integral.)

\(^1\)This series of single integrals referred to as an “iterated integral”. The power of Fubini’s theorem is that it allows multiple integrals to be evaluated via repeated applications of the Fundamental Theorem of Calculus (Calc. I). Without Fubini’s theorem, evaluation of multiple integrals would require determining the limit of a Riemann sum.
Applications of double and triple integrals include:

- the area of planar regions and volume of three-dimensional regions;
- the average value of a function over a planar or three-dimensional region;
- the mass, first moments and centers of mass, and second moments (moments of inertia) of flat plates and three-dimensional objects.

You are expected to be able to:

* set up and evaluate double and triple integrals in cartesian, polar/cylindrical and spherical coordinates; sketch the region of integration corresponding to a given iterated integral and change the order of integration for a given iterated integral;
  (suggested problems: sec 14.2 # 77, 78, 80; sec 14.4 # 20, 25, 26, 43;
  sec 14.5 # 21, 45, 48; sec 14.7 # 5, 13, 14, 25)

* use double and triple integrals to determine the area or volume of a given region;
  (suggested problems: sec 14.2 # 59, 62, 68; sec 14.3 # 15, 17; sec 14.5 # 27, 29, 36; sec 14.7 # 12, 39, 40, 49, 52, 53, 55, 57, 62)

* use double and triple integrals to determine the average value of a function over a given region;
  (suggested problems: sec 14.3 # 20–22; sec 14.4 # 33, 36, 45; sec 14.5 # 39; sec 14.7 # 66)

* use double and triple integrals to find the mass, first and second moments and center of mass of a two- or three-dimensional object;
  (suggested problems: sec 14.6 # 9, 10, 16, 17, 19, 22, 23, 25; sec 14.7 # 67, 70, 80, 82)

* given a coordinate transformation \( x = x(u, v), y = y(u, v) \), set up and evaluate a double integral in the \( xy \)-plane using the change of coordinates formula (this includes computing the Jacobian of the coordinate transformation and finding the corresponding region of integration in the \( uv \)-plane); given coordinate functions \( u = u(x, y), v = v(x, y) \), solve for the corresponding coordinate functions \( x = x(u, v), y = y(u, v) \).
  (suggested problems: assignment 10).
(SCALAR) LINE INTEGRALS

Recall:

* A curve is smooth if there is a position function \( \vec{r}(t) \) parameterizing the curve with \( \vec{r}'(t) \) continuous and \( \vec{r}'(t) \neq \vec{0} \) — see section 12.1 for a review of parameterized curves.

* The infinitesimal vector \( d\vec{r} \) is the vector differential of the position vector \( \vec{r} \). If \( \vec{r}(t) \) is a smooth parameterization of a curve, then \( d\vec{r} \) is tangent to the curve, and \( d\vec{r} = \vec{r}'(t) \, dt \).

* The scalar line element \( ds = |d\vec{r}| \) (also called the arc length element, first introduced in sec 12.3) is a measure of distance, and can be thought of as an infinitesimal version of the Pythagorean theorem: \( ds^2 = dx^2 + dy^2 + dz^2 \). If \( C \) is a curve with a smooth parameterization \( \vec{r}(t) \), the arc length element with respect to the parameterization is \( ds = |d\vec{r}| = |\vec{r}'(t)| \, dt \).

Key Ideas:

- A scalar line integral is a generalization of integrals of a single variable. If \( C \) is a smooth curve and \( f \) is a real-valued function defined along the curve, the (scalar) line integral of \( f \) along \( C \) is \( \int_C f \, ds \). Applications of scalar line integrals include:
  - length of curves (covered in sec 12.3);
  - the mass, first moments and centers of mass, and second moments (moments of inertia) of one-dimensional objects (for example, thin wires).

You are expected to be able to:

* set up and evaluate scalar line integrals; use scalar line integrals to determine the length of a curve; use scalar line integrals to find the mass, center of mass and first and second moments of a one-dimensional object.

(suggested problems: assignment 10)