The second midterm will be held in class (10:40–11:30pm) on Friday 14 October. You will be allowed one half of one side of an 8.5″ × 11″ sheet of paper (total surface area: 8.5″ × 5.5″) for handwritten notes, the table of basic derivatives and integrals posted on the course website, and a scientific calculator. No other notes or technology will be allowed for this exam.

This exam will be over material covered in class since the previous exam, corresponding to chapter 13 in the text. Suggested review problems are included in this review. A series of problems followed by the phrase “as needed” indicates exercises aimed at developing basic skills. Other problems require more thought and usually highlight an important concept or application.

INTRODUCTION TO SCALAR-VALUED FUNCTIONS OF TWO OR THREE VARIABLES

Key Ideas:

- The domain of a function of two variables is the set points (ordered pairs of numbers) in \( \mathbb{R}^2 \) for which the function is defined. Similarly, the domain of a function of three variables is the set of points (ordered triples) in \( \mathbb{R}^3 \) for which the function is defined. The range of a scalar-valued function is the set of all values in \( \mathbb{R} \) taken on by the function over its domain. Unless otherwise stated, the functions under discussion here are assumed to be scalar-valued functions of two or three variables.

- The graph of a function of two variables (also called the surface generated by the function) is the set of a two-dimensional surface in \( \mathbb{R}^3 \); it can provide a useful visual representation of the function. The graph of a function of three variables is a three-dimensional object sitting in \( \mathbb{R}^4 \); it does not provide a useful visual representation of the function.

- The level sets of a function are the set of points in its domain on which the function value remains constant. If \( f \) is a function of two variables, the level set \( f(x, y) = c \) is a curve (or collection of curves) in the \( xy \)-plane. If \( f \) is a function of three variables, the level set \( f(x, y, z) = c \) is a surface (or collection of surfaces) in \( \mathbb{R}^3 \). Level sets provide a visual representation for a function of two and three variables, which can be used to analyze the behavior of the function. They can also be used to define curves and surfaces.
You are expected to be able to:

* sketch and find the equations of level sets of a given function;
  (suggested problems: sec 13.1 # 15, 16, 59, 61)

* match combinations of the level curves of a function, the surface generated
  by the function and the equation of the function;
  (suggested problems: sec 13.1 # 31–36)

* identify points of discontinuity in the domain of the function. (suggested
  problems: sec 13.2 # 31–40 as needed)

DIFFERENTIABLE FUNCTIONS OF TWO OR THREE VARIABLES

Key Ideas:

- Geometrically, the derivative of a function of a single variable is the slope
  of the tangent line to the graph of the function. For a function of more
  than one variable, there are potentially infinitely many lines tangent to
  its graph. This leads to the question: what does it mean for a function
  of more than one variable to be differentiable? For our purposes, we will
  say that a function of more than one variable is differentiable at a point
  if there is a well-defined tangent space at that point. The dimension of
  the tangent space agrees with the dimension of the domain; for example,
  a function of two variables is differentiable at a point if it has a tangent
  plane at that point.

- The tangent space to a function $f$ is the graph of a linear function $L$, called
  the local linearization of $f$. Of all linear functions, $L$ is the one that best
  approximates $f$. Informally, if you “zoom in” on the point of tangency, the
  graphs of $f$ (the surface generated by $f$) and of $L$ (the tangent space) look
  the same. The linearization/equation of the tangent space of a function
  at a point can be used to approximate the function’s values near that point.

- If a function $f$ is differentiable, its differential $df$ reflects the change in
  $f$ over an infinitesimal displacement $d\vec{r}$ in the domain. Differentials are
  sometimes used to approximate the change in the value of a function over
  small (but not infinitesimal) displacements $\Delta \vec{r}$ in the domain. Note: the
  differential is not a derivative! A derivative is a rate of change of the
  function; the differential is a change in the function value.
• As with functions of a single variable, derivatives of functions of more than one variable measure instantaneous rates of change of the function. With functions of several variables, however, the rate of change depends on the path traveled in the domain.

  – A partial derivative is the instantaneous rate of change of a function with respect to one variable as the others are held constant. Geometrically, it is the slope of the tangent line to the graph of the function when the function is restricted to a path in the domain parallel to a coordinate axis. Computationally, the partial derivative is a Calc. I derivative taken with respect to the variable that is allowed to change, while treating the remaining variables as constants. If the partial derivatives of a function are continuous on an open region, then the original function is differentiable on that region. Higher-order partial derivatives of a function are defined by taking successive partial derivatives. If second-order partial derivatives are continuous on an open region, then the order of differentiation does not matter when computing second-order partial derivatives on this region. The units of a partial derivative will be the units of the function values over the units of the variable with respect to which the derivative is taken (eg: if the function gives the temperature in degrees celsius and the units in the domain are distance measured in meters, then the units of the partial derivatives will be °C/m).

  – A directional derivative generalizes partial derivatives to straight-line paths in the domain. A directional derivative is the instantaneous rate of change of a function with respect to a given direction in its domain. Geometrically, a directional derivative is the slope of the line tangent to the graph of the function when the function is restricted to a line in the domain. The units of a directional derivative will be the units of the function values over the units of the domain (eg: if the function gives the temperature in degrees celsius and the units in the domain are distance measured in meters, then the units of the directional derivatives will be °C/m).

  – The chain rule measures the instantaneous rate of change of a function with respect to a parameterization. Geometrically, it is the slope of the line tangent to the graph of the function when the function is restricted to a parameterized curve in the domain, times the speed of travel along that curve. The units of the chain rule are the units of the function values over the units of the parameter (eg: if the function gives the temperature in degrees celsius and the units of the parameter is time measured in seconds, then the units of the chain rule will be °C/s).
• The gradient of a differentiable function is a vector field whose components are the partial derivatives of the function. The gradient has several useful properties:

  – The direction of the gradient at a point is the direction in which the function increases most rapidly, and the magnitude of the gradient gives the maximum rate of increase of the function in any direction (that is, it gives the maximum value among all directional derivatives). Similarly, the function decreases most rapidly in the direction opposite that of the gradient, and the negative of the magnitude of the gradient gives the minimum value of the directional derivatives of the function.

  – The gradient of a function is orthogonal to the function’s level sets. This fact can be used to find equations of tangent and normal lines to level curves or tangent planes and normal lines to level surfaces.

• The differential, the directional derivatives and the chain rule can all be expressed in terms of the dot product of the gradient with an appropriate vector in the domain:

  \[
  \text{differential: } \quad df = \nabla f \cdot dr \\
  \text{directional derivative: } \quad D_uf = \nabla f \cdot \hat{u} \\
  \text{chain rule: } \quad \frac{df}{dt} = \nabla f \cdot \vec{v}
  \]

  \[
  dr = dx \hat{i} + dy \hat{j} + dz \hat{k} \\
  \hat{u} \text{ a unit vector} \\
  \vec{v} \text{ the velocity to the curve } \vec{r}(t)
  \]

• Summary: the local linearization/equation of the tangent space of a function is used to approximate the function’s values. The differential of a function is used to approximate changes in the function’s values. Derivatives of a function measure the function’s rates of change: partial derivatives measure rates of change in directions parallel to the coordinate axes; directional derivatives measure rates of change in arbitrary directions in the domain; the chain rule measures rates of change relative to a parameterization in the domain.

You are expected to be able to:

* compute partial derivatives of any order;
(suggested problems: sec 13.3 # 1–49 as needed, # 56, 65, 67)

* compute the gradient; apply the properties of the gradient to find directions and values of fastest increase and decrease of a function, to find directions of zero change, and to find equations of normal and tangent
lines to level curves; compute directional derivatives; use the directional
derivative to determine the rate of change of a function in a given direc-
tion; understand the relationship between the gradient and the directional
derivative; (suggested problems: sec 13.5 # 1–10 as needed, # 11–18 as
needed, # 19–24 as needed, # 25, 28, 32, 34–36)

* compute the chain rule; use the chain rule to determine the (instantaneous)
rate of change of a function along a parameterized curve; use the chain
rule, possibly in combination with given numerical data, to determine how
a function of several variables changes with respect to a parameter;
(suggested problems: sec 13.4 #1–8 (using chain rule only) as needed, 41,
42, 47, 48)

* determine whether the chain rule or a directional derivative is the correct
type of derivative to use in a given context;
(suggested problems: sec 13.6 # 23)

* find the equation of the tangent plane to a surface;
(suggested problems: sec 13.6 # 11, 12)

* compute the linearization of a function at a point; use the linearization to
estimate function values near the point;
(suggested problems: sec 13.6 # 31, 32)

* compute the differential; use the to estimate changes of the function value
over small distances in the domain; use the differential to determine how
small perturbations of the input variables will affect the output of the
function (that is, determine the sensitivity of the function to small per-
turbations in the domain).
(suggested problems: sec 13.6 # 19, 51, 52, 54)

LOCAL AND ABSOLUTE EXTREMA
&
OPTIMIZATION SUBJECT TO CONSTRAINT

Key Ideas:

- Local maxima and minima of a function of two variables occur at critical
  points; that is, points where either both partial derivatives are zero, or
  where one or both of the partial derivatives fail to exist. As with func-
tions of a single variable, not every critical point is an extrema; saddle
  points are also possible. The second derivative test can often be used to
classify the critical points as maxima, minima or saddle point. Be aware
that there are instances where the second derivative test either cannot be used (for example, \( f(x, y) = \sqrt{x^2 + y^2} \)) or is inconclusive.

- To find the absolute maxima and minima of a function of two variables over a closed and bounded region, first find all critical points in the interior of the region, then find all critical points on the interior of the boundary curves. Evaluate these critical points together with the endpoints of the boundary curves and choose those with the largest and smallest values.

- The method of Lagrange multipliers is used to detect extrema of a function of two or three variables subject to some constraint of the domain.

You are expected to be able to:

* locate the critical points of a function of two variables; recognize when the second derivative test can be used to classify local extrema of a function of two variables, and be able to carry out the test;  
  (suggested problems: sec 13.7 # 3, 8, 21)

* identify the absolute maximum and minimum values and the points at which they occur for a function of two variables on a closed and bounded region;  
  (suggested problems: sec 13.7 # 31)

* use the method of Lagrange multipliers to identify the absolute maximum and minimum values and the points at which they occur for a function of two or three variables subject to a single constraint.  
  (suggested problems: sec 13.8 problems from assignment 7)