The Fibonacci Sequence

What is it? The Fibonacci Sequence is a sequence of numbers such that each number is the sum of the two previous starting with 0 and 1. In algebraic form:

\[ F_n = F_{n-1} + F_{n-2} \]

where \( F_0 = 0 \) and \( F_1 = 1 \). This says more explicitly that the \( n \)th number in the sequence (represented by \( F_n \)) is the sum of the two before starting with 0 and 1.

Fibonacci Numbers in Modular Arithmetic

\[ a \mod b = \tau + qb \mod b \]

. Modular arithmetic is all about remainders.

\[ a = \tau + qb \]

Where \( a \) and \( b \) are integers, \( q \) the quotient of \( a/b \), \( \tau \) the remainder of \( a/b \). However, \( qb \mod b \) equals zero because a multiple of a dividend does not change the remainder. This gives:

\[ a \mod b = \tau \]

The Fibonacci numbers are calculated using modular arithmetic where the “\( a \)” is another Fibonacci number (\( \mod F_b \)). Some examples are:

- \( \mod 8: 0, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, ... \)
- \( \mod 13: 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ... \)
- \( \mod 21: 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ... \)

When the Fibonacci Numbers are evaluated this way a "cycle” pattern emerges.

Cycles and Partial Cycles

When looking at Fibonacci numbers \( \mod b \) something this paper will refer to as a "cycle” appears. A cycle is defined as the complete sequence of numbers before repetition. From there, a “partial cycle” is defined as the terms between each 0 of the cycle. For example: The Fibonacci numbers in \( \mod 8 \) are 0,1,1,2,3,5,0,5,5,2,7,1, finding the Pisano Period length for a period created by \( F_2 \) instead of \( F_1 \) means there are two partial cycles which are 0,1,1,2,3,5 and 0,5,5,2,7,1. These partial cycles make up a full cycle which would then continuously repeat. These partial cycles will always present themselves in one of two ways. First, it may be made up of quarter cycles where there are four sequences separated by zeros. This cycle may also be made of half cycles where there are only two sequences of numbers separated by zeros. It can be seen why and when this happens. First, a description of how these cycles appear. In the first table for example shows the Fibonacci numbers \( \mod 13 \). This table has been strategically shown as a full cycle and each row as a partial cycle.

Table: Table of some of the Fibonacci numbers \( \mod 13 \)

One can take that full cycle and begin to generalize it. First, writing the first partial cycle as the corresponding term numbers \( F_n \). Then, the second partial cycle can be written using sums because the Fibonacci sequence is recursive using addition. This is illustrated in the second table. \( F_1 \) through \( F_7 \) is explicitly stated and the consecutive terms are all written as sums of \( F_1 \)’s and \( F_2 \)’s.

Table: Generalizing sums of \( F_1 \)

The fact that any factor of \( F_2 \mod F_2 = 0 \) can then be used. In the next table, any factor of \( F_2 \) is reduced to 0. This also affects the sums of the other terms. With the first term of each partial cycle now being a 0, the coefficients on the \( F_2 \)’s are significantly smaller.

Table: Simplification of previous table.

It is now possible to generalize this cyclic pattern using \( F_n \) terms. The first row is the Fibonacci numbers \( F_3 \) through \( F_{n-1} \). The second partial cycle has been converted from the previous table to reflect the fact that the coefficients are factors of \( F_3 \)’s or in general, \( F_n \)’s. The third partial cycle also shows this but as the row number increases, the number of factors of \( F_1 \)’s does as well. In all terms, there are factors of the Fibonacci numbers as well.

Table: Simplified generalized partial cycles of Fibonacci numbers in modular arithmetic

From this now fully generalized pattern, one can now begin to understand when and why cycles are made up of only quarter or half cycles. The last term of a cycle is always a 1. This can be used to test if a cycle has quarter or half partial cycles. One can test the last term of the second cycle \( \mod F_n \) to see if it evaluates as a 1. If it does, then there are two partial cycles. If not, then there must be four partial cycles. One could also prove that these are the only options (half or quarter partial cycles) using Cassini’s identity which states:

\[ F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1} \]

Yet \( F_n^2 \) is 0 because it is a factor of \( F_n \). So,

\[ F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1} \]

Which can be reduced to

\[ F_{n-1}F_{n+1} = (-1)^{n} \]

and then continued

\[ F_{n+1}^2 = -1^n \]

There are then only two options. One when \( n \) is even which offers that

\[ F_{n-1}^2 = 1 \]

and when \( n \) is odd and

\[ F_{n+1}^2 = 1 \]

Therefore, \( \mod F_2 \) alternates between two partial cycle lengths. This will be used in the proof that these partial cycles happen with either quarter or half cycles.

Another Interesting Thing

"Multiples" The first thing to note is that multiples of numbers seem to appear in a pattern similar to that of the normal number system like every other number being a multiple of 2 ("even numbers") and every fifth number being a multiple of 5. The Fibonacci sequence has a similar structure demonstrate. First there are some assumptions: the definition of an odd number is \( 2n+1 \), the definition of an even number is \( 2n \), that \( odd + odd = even \), that \( even + odd = odd \), and \( F_2 = F_0 \). \( F_1 = 1 \). Use this to build the following sequence.

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, ... \]

Replace the first term with \( 2n + 1 \) while keeping the zeroth term 0. This makes a new sequence: \( 0, 2n+1, 2n+1, 2(2n+1), 3(2n+1), 5(2n+1), 2 \times 4(2n+1), 13(2n+1), 3 \times 7(2n+1), ... \)

This suggests that depending on what \( 2n + 1 \) equals, every number is a multiple of \( 2n + 1 \). This means one can assume that if \( 2n + 1 \) is replaced with \( 2n \) the same conclusion is going to hold. The multiples appear in a pattern. Since 2 is the third term, every third number will be even. Following this pattern for other terms: every \( 3n \)th term is going to be a multiple of \( k \).

Pisano Periods

A Pisano period is the length of a full cycle for any \( \mod n \) as long as \( n \) is a real number. The Fibonacci numbers \( \mod D \) are \( 0, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, ... \) This shows a period length of 16.

Pisano Periods \( \mod F_n \). The Pisano period length for a period created by \( \mod F_n \) is the number of partial cycles times the term number. Examine \( \mod 8 \): The term number is 6 and has two partial cycles. \( \Rightarrow 6 \times 2 = 12 \). Examine \( \mod 13 \): The term number is 7 and has four partial cycles. \( \Rightarrow 7 \times 4 = 28 \) This means that for term number \( n \):

- For all \( n \) even numbers, the Pisano period length is \( 2n \).
- For all \( n \) odd numbers, the Pisano period length is \( 4n \).