In these problems we take an algebraic approach to computing tangent-line slopes.

The homework problems which follow give you a function \( f \) for which you are to compute \( m_{\text{tan}} \), the slope of the line tangent to the graph of \( f \) at the point where \( x = A \). We present examples (A) and (B). The assigned problems follow on the last page.

(A) The function \( f(x) = \frac{x}{x + 1} \) has appeared in some of our previous homework.

**STEP ONE:** We begin calculating \( m_{\text{tan}} \) by laying out and simplifying the difference-quotient expression for \( m_{\text{seq}} \):

\[
m_{\text{seq}} = \frac{f(A + h) - f(A)}{h} = \frac{h}{(A + h)(A + 1) - A(A + h + 1)} = \frac{h}{(A + h)(A + 1) - A(A + h + 1)} = \frac{h(A + h + 1)(A + 1)}{(A^2 + Ah + A + h) - (A^2 + Ah + A)} = \frac{1}{(A + h + 1)(A + 1)}.
\]

Here you can see that we have simplified things to the point where we were able to cancel out the starting denominator’s \( h \) factor.

We now have a formula for \( m_{\text{seq}} \), the slope of the line connecting the point of tangency \( P = (A, f(A)) \) to the variable point \( Q = (A + h, f(A + h)) \).

**STEP TWO:** Now we finish off the \( m_{\text{tan}} \) calculation by bringing point \( Q \) closer and closer to the point of tangency \( P \). We do this by imagining what happens to

\[
m_{\text{seq}} = \frac{1}{(A + h + 1)(A + 1)}
\]

as the number \( h \) gets closer and closer to 0: as \( h \) gets close to 0, \( (A + h + 1) \) gets closer and closer to \( (A + 1) \).
And thus the denominator of the $m_{sec}$ expression gets closer and closer to $(A + 1)^2$ as $h$ gets closer and closer to 0. Thus

$$m_{tan} = \frac{1}{(A + 1)^2}.$$

We can summarize our work by saying that as $h \to 0$ we have

$$m_{sec} = \frac{1}{(A + h + 1)(A + 1)} \to m_{tan} = \frac{1}{(A + 1)^2}.$$

We can use this formula to write tangent-line equations:

(a) If the point of tangency is at $x = 0$, then $A = 0$ and we have $f(A) = 0$, and hence point of tangency $P = (A, f(A)) = (0, 0)$ along with $m_{tan} = \frac{1}{(A + 1)^2} = 1$.

So the tangent line here is the line through $(0, 0)$ with slope 1: $y = x$.

(b) If the point of tangency is at $x = -1/2$, then $A = -1/2$ and we have $f(A) = -1$, and hence point of tangency $P = (A, f(A)) = (-1/2, -1)$ along with $m_{tan} = \frac{1}{(A + 1)^2} = 4$.

So the tangent line here is the line through $(-1/2, -1)$ with slope 4: $y = -1 + 4(x + 1/2)$ or $y = 4x - 3$.

(B) Here we use the **Alternative Difference Quotient** to come up with the line tangent to the graph of $f(x) = 5 + 8x - 3x^2$ at the point where $x = 2$.

The Alternative Difference Quotient gives $m_{seq}$ in the form of the slope of the line connecting the point of tangency $P = (A, f(A))$ with the variable point $Q = (t, f(t))$:

$$m_{seq} = \frac{f(t) - f(A)}{t - A}.$$

We bring $Q$ closer and closer to point of tangency $P$ by bringing $t$ closer and closer to $A$. 
STEP ONE: For our particular function we have, as long as \( t \neq A \),

\[
m_{\text{seg}} = \frac{f(t) - f(A)}{t - A} = \frac{(5 + 8t - 3t^2) - (5 + 8A - 3A^2)}{t - A} = \frac{8t - 3t^2 - 8A + 3A^2}{t - A} = \frac{8(t - 8A)}{t - A} + \frac{-3t^2 + 3A^2}{t - A} = 8 - 3\frac{(t + A)(t - A)}{t - A} = 8 - 3(t + A),
\]

so that, if \( t \neq A \), we have \( m_{\text{seg}} = 8 - 3(t + A) \).

STEP TWO: Now we imagine what happens to \( m_{\text{seg}} \) as \( t \) approaches \( A \): as \( t \) gets closer and closer to \( A \), \( (t + A) \) gets closer and closer to \( (A + A) \). Thus

\[
m_{\text{tan}} = 8 - 3(A + A) = 8 - 6A
\]

Now we are ready for the line tangent to the graph of \( f(x) = 5 + 8x - 3x^2 \) at the point where \( x = 2 \). We have \( A = 2 \), so the point of tangency is

\[
P = (A, f(A)) = (2, f(2)) = (2, 9)
\]

and the tangent line’s slope is

\[
m_{\text{tan}} = 8 - 3(A + A) = 8 - 6A = 8 - 6(2) = 8 - 12 = -4.
\]

This makes for tangent line equations of the form

\[
y = 9 - 4(x - 2) \quad \text{or} \quad y = -4x + 17
\]
Assigned Problems: For each of problems 1-4, use difference-quotient methods to find an expression for $m_{tan}$ for the point of tangency $(A, f(A))$.

1. $f(x) = 4x^2 - 3x$
2. $f(x) = x^3$
3. $f(x) = \frac{1}{x}$
4. $f(x) = \sqrt{x}$

5. Find an equation for the line tangent to the graph of the function in Example (B), above, at the point where $x = 4/3$.

6. Find an equation for the line perpendicular to the graph of $f(x) = 4x^2 - 3x$ at the point where $x = -2$. 