Chapter 2, 1:) Since \( A \subseteq B \) it follows \( B = A \cup (B \setminus A) \) with \( B \setminus A \in \mathcal{A} \) and \( A \cap (B \setminus A) = \emptyset \). Since \( m \) is countably additive over disjoint unions \( m(B) = m(A \cup (B \setminus A)) = m(A) + m(B \setminus A) \geq m(B) \) because \( m : \mathcal{A} \to [0, \infty] \). (Note that the inequality is trivially true if \( m(B) = \infty \), and \( m(B \setminus A) = \infty \) is only possible if \( m(B) = \infty \).)

Chapter 2, 2:) Since \( \emptyset \in \mathcal{A} \) and \( A \cap \emptyset = \emptyset \) it follows \( m(A) = m(A \cup \emptyset) = m(A) + m(\emptyset) \). It follows that \( m(\emptyset) = 0 \) because for extended real numbers \( x, y \geq 0 \), \( x + y = x \) and \( y \neq 0 \) is only possible if \( x = \infty \). (Note that if \( m(A) = \infty \) for all sets in \( \mathcal{A} \) then of course \( m(\emptyset) = \infty \) too. Such \( m \) trivially satisfies the assumptions given. Also note that very well \( \mathcal{A} = \{\emptyset\} \) is possible.)

Chapter 2, 3:) Replace \( E_k \) by \( E'_k := E_k \setminus (\cup_{j=1}^k E_j) \) for \( k \geq 2 \). Then the collection of sets \( \{E'_k\} \) is disjoint with the same union as \( \cup_{k=1}^\infty E_k \). The sets \( E'_k \) are in \( \mathcal{A} \) because \( \mathcal{A} \) is a \( \sigma \)-algebra and \( m(E'_k) \leq m(E_k) \) because of the monotonicity proven in Problem 1. It follows from countably additivity with respect to disjoint unions

\[
m(\cup_{k=1}^\infty E_k) = m(\cup_{k=1}^\infty E'_k) = \sum_{k=1}^\infty m(E'_k) \leq \sum_{k=1}^\infty m(E_k).
\]

Chapter 2, 4:) \( c \) is a set function by definition. Let \( \{E_k\} \) be a countable collection of disjoint sets. Note that a finite union of disjoint finite sets is finite and the numbers of elements add up for the disjoint union. Thus additivity holds in this case. If there is one set \( E_k \) with infinitely many elements then also \( \cup_k E_k \) has infinitely many elements, independent of whether we have a finite or countably infinite collection of sets. In this case both sides of the equation \( c(\cup_k E_k) = \sum_k c(E_k) \) are infinite and thus holds. If all sets are finite with infinitely many nonempty the set \( \cup_{k=1}^\infty E_k \) has infinitely many elements and \( c(\cup_{k=1}^\infty E_k) = \infty \). But in this case in the sum \( \sum_{k=1}^\infty c(E_k) \) also infinitely many summands \( c(E_k) \) are \( \geq 1 \) and thus the series diverges and has value \( \infty \). Thus countable additivity holds in all cases. Translation invariance follows because translation by a number \( y \) is a bijection of sets (with inverse function the translation by \( -y \)) and thus does not change cardinalities. But the cardinality of a subset of \( \mathbb{R} \) determines the value of \( c \).

Chapter 2, 5:) The outer measure of the interval \([0, 1]\) is its length and thus \( 1 \). The outer measure of each countable set is \( 0 \). Since \( 1 \neq 0 \) it follows \([0, 1]\) is not countable

Chapter 2, 6:) Let \( A \) respectively \( B \) be the set of irrationals respectively rationals in the interval \([0, 1]\). Then \( m^*(B) = 0 \) because \( B \) is a subset of a countable set and thus countable. Also \( A \cup B = [0, 1] \) and by countable subadditivity \( 1 = m^*([0, 1]) = m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(A) \). Because \( A \subseteq [0, 1] \) by monotonicity \( m^*(A) \leq m^*([0, 1]) = 1 \). Thus \( m^*(A) = 1 \).
Chapter 2, 7:) By definition of outer measure for each positive integer \( k \) there exists a countable collection of open bounded intervals \( \{I_j^{(k)}\}_j \) covering \( E \) such that \( \sum_j \ell(I_j^{(k)}) < m^*(E) + \frac{1}{k} \). Then \( \bigcup_j I_j^{(k)} \) is a union of open sets and thus open for each positive integer \( k \) and \( G := \cap_{k=1}^{\infty} \bigcup_j I_j^{(k)} \) is a \( G_\delta \)-set. For each \( k \) we have \( E \subseteq G \subseteq \bigcup_j I_j^{(k)} \). From this we get by monotonicity \( m^*(E) \leq m^*(G) \leq m^*(E) + \frac{1}{k} \) for all positive integers \( k \). It follows that \( m^*(G) = m^*(E) \).

Chapter 2, 8:) Note that the union of two intervals is disjoint or an interval. Thus \( [0,1] \setminus \bigcup_{i=1}^n I_k \) is a disjoint union of a set of finitely many points \( C = \{c_1, \ldots, c_r\} \) and intervals (To see this just write \( \bigcup_{k=1}^n I_k \) as a union of disjoint intervals \( (a_i, b_i), i = 1, \ldots, r \) with \( b_i < a_{i+1} \)). The points are irrational but actually no intervals can occur because each interval contains rational numbers. Then \( m^*(C) = 0 \) and by subadditivity and monotonicity:

\[
\sum_{k=1}^n \ell(I_k) \geq m^*(\bigcup_{k=1}^n I_k) \geq m^*([0,1]\setminus C) = m^*([0,1]\setminus C) + m^*(C) \geq m^*([0,1]) = 1
\]

Chapter 2, 14:) Given \( E \) with \( m^*(E) > 0 \) let \( E_n := E \cap (-n, n) \) for \( n \in \mathbb{N} \). Then \( E = \bigcup_{n \in \mathbb{N}} E_n \) and each \( E_n \) is bounded. By countable subadditivity

\[
0 < m^*(E) \leq \sum_{n \in \mathbb{N}} E_n.
\]

If \( m^*(E_n) = 0 \) for all \( n \in \mathbb{N} \) then the series has value 0, which is a contradiction. Thus \( E_n \subseteq E \) has outer measure > 0.

Chapter 2, 15:) First assume \( E \subseteq [-M, M] \) is bounded. Then a finite number of disjoint intervals of the form \( J_k := (k\varepsilon/2, (k + 1)\varepsilon/2] \) will cover \( [-M, M] \) (by the Archimedian property) and thus \( E \). Since \( E_k := M \cap J_k \) is measurable of measure \( \leq \varepsilon \) we have \( E \) a disjoint union of measurable sets \( E_k \) with \( m(E_k) < \varepsilon \). If \( E \) is unbounded cover \( E \) by a countably infinite collection \( (I_k) \) of bounded open intervals such that \( \sum_k \ell(I_k) < m(E) + 1 \). Since the series converges there exists \( N \) such that \( \sum_{k=1}^N \ell(I_k) < \varepsilon \). Let \( E_1 := E \cap (\bigcup_{k=N+1}^\infty I_k) \). Then \( E_1 \) is measurable of measure \( < \varepsilon \) and \( E_2 := E \setminus E_1 \subset \bigcup_{k=1}^N I_k \) is bounded and can be written by the above as a finite union of disjoint measurable sets with each of measure \( \leq \varepsilon \).

Chapter 2, 19:) It follows from Theorem 11, (i) that \( E \) not measurable implies that there exists \( \varepsilon > 0 \) such that for each open set \( O \supseteq E \), \( m^*(O \setminus E) \geq \varepsilon \). There exists a countable collection of open bounded intervals \( \{I_k\} \) such that \( \sum_k \ell(I_k) < m^*(E) + \varepsilon \). Then \( O := \bigcup_k I_k \supseteq E \) is an open set and \( m^*(O) \leq \sum_k \ell(I_k) < m^*(E) + \varepsilon \), which implies \( m^*(O) - m^*(E) < \varepsilon \). Thus \( m^*(O \setminus E) \geq \varepsilon > m^*(O) - m^*(E) \) and thus \( m^*(O \setminus E) > m^*(O) - m^*(E) \).

Chapter 2, 22:) Since \( O \supseteq A \) implies \( m^*(O) \geq m^*(A) \) it follows that \( m^*(A) \leq \inf\{m^*(O)|O \supseteq A, \ O \text{ open}\} = m^{**}(A) \). If \( m^*(A) = \infty \) the other inequality is
obvious. Otherwise, for each $\varepsilon > 0$ there exists an open set $O \supseteq A$, namely a countable union of open intervals, such that $m^*(O) < m^*(A) + \varepsilon$. Thus $m^{**}(A) \leq m^*(A) + \varepsilon$ for all $\varepsilon > 0$, which implies $m^{**}(A) \leq m^*(A)$ and thus proves $m^{**}(A) = m^*(A)$.

**Chapter 2, 23:** If $F \subseteq A$ then $m^*(F) \leq m^*(A)$ and thus $m^{***}(A) = \sup\{m^*(F) | F \subseteq A, F \text{ closed}\} \leq m^*(A)$. If $m^{***}(A) = \infty$ or $m^*(A) = 0$ then the other inequality holds obviously. In particular this is true if there exists $F \subseteq A$ closed with $m^*(F) = \infty$. So we can assume that no such $F$ exists. If $A$ is measurable then by Theorem 11, (iv) for each $\varepsilon > 0$ there exists $F \subseteq A$ closed such that $m^*(E \setminus F) = m^*(E) - m^*(F) < \varepsilon$, or $m^*(F) > m^*(E) - \varepsilon$ (note that excision works since $F$ is closed and thus measurable). Thus we can make the values $m^*(F)$ as close as we want to $m^*(E)$. It follows that $m^{***}(E) = m^*(E)$. Now assume that $E$ is not measurable. By Theorem 11, (iv) there exists $\varepsilon > 0$ such that for each closed subset $F \subseteq E$ we have $m^*(E \setminus F) = m^*(E) - m^*(F) \geq \varepsilon$. Thus $m^*(F) \leq m^*(E) - \varepsilon$ with $\varepsilon > 0$ fixed. Thus also $m^{***}(E) \leq m^*(E) - \varepsilon < m^*(E)$ in this case. (For the proof of the equivalency (i) and (iv) use that if $O \supseteq E^c$ is open then $O^c \subseteq E$ is closed and $O \cap E^c = E \setminus O^c$.)

**Chapter 2, 24:** If $E_1, E_2$ are measurable then all sets $E_1 \cup E_2, E_1 \cap E_2$, $E_2 \setminus (E_1 \cap E_2)$ are measurable. If $m(E_1 \cap E_2) < \infty$ we get from additivity of the measure and excision: $m(E_1 \cup E_2) = m(E_1 \cup (E_2 \setminus E_1)) = m(E_1) + m(E_2 \setminus E_1) = m(E_1) + m(E_2 \setminus (E_1 \cap E_2)) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$. If $m(E_1 \cap E_2) = \infty$ then by monotonicity of the measure we get $m(E_1) = \infty$ and $m(E_2) = \infty$ and thus $m(E_1) + m(E_2) = \infty = m(E_1 \cup E_2) + m(E_1 \cap E_2)$.

**Chapter 2, 30:** If $C_E$ is the choice set for the rational equivalence relation on a set $E$ then $E \subseteq \bigcup_{a \in \mathbb{Q}}(C_E + a)$ and thus is a countable union of countable sets. Thus $E$ is countable and has outer measure $0$ contradicting the assumption that $E$ has positive outer measure.

**Chapter 2, 33:** The argument for Problem 7, see also Problem 18, only required $m^*(E) < \infty$ and not the stronger assumption $E$ bounded. So there exists a $G_\delta$-set $G \supseteq E$ such that $m^*(G) = m^*(E)$. If $m^*(G \setminus E) = 0$ then $E$ would be measurable by Theorem 11 (ii). Thus $m^*(G \setminus E) > 0$.

**Chapter 2, 38:** Let $f : [a, b] \to \mathbb{R}$ be a Lipschitz-map with constant $c > 0$. We show first that $f$ maps a set $E$ of measure zero to a set of measure zero. Let $\varepsilon > 0$. An open bounded interval $I \subset \mathbb{R}$ is mapped to a set of diameter $\leq c\ell(I)$ (i.e. any two points in $f(I)$ have distance $\leq c\ell(I)$). Thus $f(I)$ is contained in some interval of length $\leq c\ell(I)$. Cover $E$ by countably many open bounded intervals $\{I_k\}$ with $\sum_k \ell(I_k) < \frac{\varepsilon}{c}$. For each $k$ choose an open bounded interval $J_k \subset \mathbb{R}$ such that $J_k \subseteq f(I_k)$ and $\ell(J_k) \leq c\ell(I_k)$. Then $\{J_k\}$ is a collection of open bounded intervals covering $f(A)$ and $\sum_k \ell(J_k) \leq \sum_k c\ell(I_k) < \varepsilon c/c = \varepsilon$. Thus $f(A)$ has measure zero. Next we show that $f$ maps $F_\sigma$-sets to $F_\sigma$-sets. Since $[a, b]$ is compact and $f$ is continuous, $f$ maps closed sets to closed sets. Since $f(\bigcup_i A_i) = \bigcup_i f(A_i)$ for all unions of sets $A_i$ it follows that $f$ maps $F_\sigma$-
sets to $F_{\sigma}$-sets. [Remark: The condition of boundedness of the domain is not necessary to see that the continuous map $f$ maps $F_{\sigma}$-sets to $F_{\sigma}$-sets. Each $F_{\sigma}$-set can also be written as a countable union of compact sets because each closed set is a countable union of compact sets by the Heine Borel theorem (just intersect the set with closed intervals $[-n, n]$). Since $f$ maps compact sets to compact sets it will map $F_{\sigma}$-sets to $F_{\sigma}$-sets. But note that Lipschitz maps do not necessarily map closed sets to closed sets: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \arctan(x)$ is Lipschitz because it has derivative $\frac{1}{1+x^2}$, which is bounded, but it maps the closed set $\mathbb{R}$ onto the set $(-\frac{\pi}{2}, \frac{\pi}{2})$, which is not closed in $\mathbb{R}$.]

Finally by Theorem 11, (iv) we know that each measurable set $E$ contains an $F_{\sigma}$-set $F$ such that $m^*(E \setminus F) = 0$. Then $f(E) = f(F) \cup f(E \setminus F)$. But both $f(F)$ and $f(E \setminus F)$ are measurable by the previous two observations, and thus $f(E)$ is measurable since a union of measurable sets is measurable.

**Chapter 2, 45:** A strictly increasing function is one-to-one and thus a bijection onto its image. Let $f : I \to f(I)$ be strictly increasing on some interval $I$. Claim: $f^{-1} : f(I) \to I$ is continuous. Note that continuity means continuity at each point. Thus it suffices to show that $f^{-1}([c, d]) : [c, d] \to [a, b]$ with $a = f^{-1}(c)$ and $b = f^{-1}(d)$ is continuous for all $c, d \in f(I)$. But $f[a, b]$ maps closed sets to closed sets (because closed subsets of $[a, b]$ are compact and continuous maps map compact sets to compact sets, and compact sets are closed). Thus the inverse image of a closed set under $f^{-1}([c, d])$ is closed and $f^{-1}([c, d])$ and thus also $f^{-1}$ is continuous.

**Chapter 2, 46:** Let $f : \mathbb{R} \to \mathbb{R}$. Let $A_f := \{E \subseteq \mathbb{R} | f^{-1}(E) \text{ is Borel} \} \subseteq 2^\mathbb{R}$.

Claim: $A_f$ is a $\sigma$-algebra containing the open sets. (This implies that $f^{-1}(B)$ is Borel ($\iff B \in A_f$) for each Borel set $B$ because the collection of Borel sets is the smallest $\sigma$-algebra containing the open sets and thus $A_f$ contains all Borel sets.) Since $f$ is continuous the inverse images of open sets are open, and thus $A_f$ contains all open sets. In particular $\mathbb{R} \in A_f$ since $f^{-1}(\mathbb{R}) = \mathbb{R}$ is Borel. Let $E \in A_f$ and thus $f^{-1}(E) \text{ is Borel}$. Then $\mathbb{R} \setminus f^{-1}(E) = f^{-1}(\mathbb{R} \setminus E)$ is Borel and thus $\mathbb{R} \setminus E \in A_f$. Let $(E_k)$ be a countable sequence of sets in $A_f$ so that $f^{-1}(E_k)$ is Borel for all $k$. Then $f^{-1}(\cup_k E_k) = \cup_k f^{-1}(E_k)$ is also Borel since the Borel sets form a $\sigma$-algebra. Thus $\cup_k E_k \in A_f$. This proves the claim. (If $f$ is defined on an interval $E$ only Borel set has to be interpreted as Borel set in $E$, i.e. as the smallest $\sigma$-algebra containing all open subsets of $E$. The result is true in this case too.)

**Chapter 2, 47:** By Problem 45 the inverse function is continuous, and thus the inverse images of Borel sets under the inverse functions are Borel sets. But this precisely means that the function itself maps Borel sets to Borel sets.

**Chapter 3, 6:** Note that the domain of $g$ is the measurable set $\mathbb{R}$. The following set equalities are checked from the definition of $g$. For $c \geq 0$ it follows that $\{x \in \mathbb{R} | g(x) > c\} = \{x \in D | f(x) > c\}$. If $c < 0$ then $\{x \in \mathbb{R} | g(x) > c\} = \{x \in D | f(x) > c\} \cup (\mathbb{R} \setminus D)$ and $\{x \in D | f(x) > c\} = \{x \in \mathbb{R} | g(x) > c\} \cap D$. If $f$ is measurable then $g$ is measurable because $\mathbb{R} \setminus D$ is measurable and unions
of measurable sets are measurable. If \( g \) is measurable then \( f \) is measurable because \( D \) is measurable and intersections of measurable sets are measurable. [Alternatively: By Proposition 5 (ii) we know that \( g \) is measurable if and only if \( g[D \setminus D, which is the function constant zero, are measurable. Because a constant function is measurable the equivalence follows.]

Chapter 3, 7:) We first show that for each function \( f : E \to \mathbb{R} \) with measurable domain \( E \) the collection \( \mathcal{B}_f \) of sets \( A \subseteq \mathbb{R} \) such that \( f^{-1}(A) \subseteq E \) is measurable is a \( \sigma \)-algebra. Note that \( f^{-1}(\mathbb{R}) = E \), which is measurable by assumption. If \( A \in \mathcal{B}_f \) then \( f^{-1}(A) \) is measurable. Thus \( E \setminus f^{-1}(A) = f^{-1}(E \setminus A) \) is measurable because complements of measurable sets in measurable sets are measurable. Thus \( \mathbb{R} \setminus A = A^c \in \mathcal{B}_f \). Finally let \( \{A_k\} \) be a countable collection of sets in \( \mathcal{B}_f \). Then \( f^{-1}(A_k) \) is measurable for each \( k \). Thus \( \bigcup_k f^{-1}(A_k) = f^{-1}(\bigcup_k A_k) \) is measurable because unions of measurable sets are measurable. Thus \( \bigcup_k A_k \in \mathcal{B}_f \).

Now suppose that \( f \) is measurable. Then inverse images of open sets are measurable by Prop. 2 and thus \( \mathcal{A}_f \) contains the open sets. Because it is a \( \sigma \)-algebra containing the open sets it contains all Borel sets because the Borel algebra is the smallest \( \sigma \)-algebra containing the open sets. Thus \( \mathcal{A}_f \) contains the Borel algebra. This means that the inverse image of each Borel set is measurable. Conversely, each open set \( U \) is a Borel set. Because \( f^{-1}(U) \) is measurable for all open sets \( U \) it follows that \( f \) is measurable by Prop. 2.

Chapter 3, 11:) From the remark about the solution to Problem 38 in Chapter 2 above it follows that the result that a Lipschitz function maps measurable sets to measurable sets also holds for Lipschitz functions with domain \( \mathbb{R} \). Thus for \( U \subseteq \mathbb{R} \) open we have \( f^{-1}(U) \) measurable by Proposition 2. Thus \( g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U) \) is measurable. Then again by Proposition 2 it follows that \( f \circ g \) is measurable. [More directly without reference to the solution of Problem 38 we can argue as follows (but are in fact using the same argument): \( f^{-1}(U) \) is measurable and thus \( f^{-1}(U) = V \cup W \) with \( V \) an \( \mathcal{F}_\sigma \)-set and \( W \) a set of measure 0 (Theorem 11 in Chapter 2). Then we know by the proof given for Problem 38 that \( g^{-1} \) maps sets of measure 0 to sets of measure 0 (this part of the argument did not require compactness of the domain). Since \( g^{-1} \) is continuous it maps \( \mathcal{F}_\sigma \)-sets to \( \mathcal{F}_\sigma \)-sets. Thus \( g^{-1}(V \cup W) = g^{-1}(V) \cup g^{-1}(W) \) is the union of a set of measure 0 and an \( \mathcal{F}_\sigma \)-set and thus is measurable.]

Chapter 3, 21:) Let \( f := \inf\{f_n\} \). Note that \( f(x) < c \iff f_n(x) < c \) for some \( n \). Thus for all \( c \in \mathbb{R} \), \( \{x \in E \mid f(x) < c\} = \cup_n \{x \in E \mid f_n(x) < c\} \) \( \supseteq \) follows from \( f \leq f_n \) for all \( n \), and \( \subseteq \) follows because if \( f(x) < c \) then there exists some \( n \) such that \( f_n(x) < c \) by the definition of \( \inf \). \( f \) is a countable union of measurable sets and thus measurable. It follows by Proposition 1 that \( f \) is measurable. By Theorem 6 since \( f_n \) is measurable also \( -f_n \) is measurable. Thus \( \inf\{-f_n\} \) and by Theorem 6 again also \( -\inf\{-f_n\} = \sup\{f_n\} \) is measurable. Finally \( \lim \inf \{f_n\} = \inf_n \inf_{k>n} \{f_k\} \) and \( \lim \sup \{f_n\} = \inf_n \sup_{k>n} \{f_k\} \) are measurable. (See Definition 19 in Chapter 1 for the definition of \( \lim \inf \) and \( \lim \sup \) and note that the sequence of sets \( A_n := \{a_k \mid k \geq n\} \) is a decreasing sequence of subsets of \( \mathbb{R} \) and thus \( \inf A_k \) respectively \( \sup A_k \) is an increasing
Chapter 3, 22:) Note that for \( f_n \) increasing with \( f_n(x) \to f(x) \) for \( n \to \infty \) it follows that \( |f(x) - f_n(x)| = f(x) - f_n(x) \). For each \( \varepsilon > 0 \) let 
\[ E_n := \{ x \in E | f(x) - f_n(x) < \varepsilon \} \]. Then \{\( E_n \}\} is a covering of \( E \) and the sequence of sets (\( E_n \)) is increasing: If \( f(x) - f_n(x) < \varepsilon \) then also \( f(x) - f_{n+1}(x) < \varepsilon \), and if \( x \in E \) then because of convergence of \( f_n(x) \) to \( f(x) \) there exists some \( N \) such that \( f(x) - f_N(x) < \varepsilon \) and thus \( x \in E_N \). Since \( f - f_n \) is continuous for each \( n \), it follows that \( E_n = (f - f_n)^{-1}(-\infty, \varepsilon) = (f - f_n)^{-1}(\varepsilon, \infty) \) is open, being the preimage of an open set under a continuous function. Thus the covering \{\( E_n \}\} is an open covering. By compactness of \([a, b]\) there is a finite subcovering \{\( E_{k_1}, \ldots, E_{k_r} \}\}. If \( N := \max(k_1, \ldots, k_r) \) then \( E_N \supseteq [a, b] \). Thus \( |f(x) - f_n(x)| < \varepsilon \) for all \( x \in E \) and all \( n > N \) (where we use that \( f(x) - f_n(x) \leq f(x) - f_N(x) \) for \( n \geq N \)).

Chapter 3, 23:) Let \( f^+ := \max(f, 0) \) and \( f^- := -\max(-f, 0) \). Then both \( f^+ \geq 0 \) are measurable by Proposition 8, and \( f = f^+-f^- \), also \( |f| = f^++f^- = |f^+|+|f^-| \). By the simple approximation theorem we can find sequences \( \varphi_n^\pm \) of simple functions such that \( \varphi_n^\pm \to f^\pm \) pointwise on \( E \) and \( 0 \leq \varphi_n^\pm \leq f^\pm \). Then \( \varphi_n := \varphi_n^+ - \varphi_n^- \) is a sequence of simple functions converging pointwise to \( f = f^+-f^- \), and \( |\varphi_n| = |\varphi_n^+ - \varphi_n^-| \leq |\varphi_n^+|+|\varphi_n^-| \leq f^+ + f^- = |f| \). This construction coincides with the old construction from the proof of the Simple Approximation Theorem, and thus concludes the proof.

Chapter 3, 24:) Let \( g \) be a strictly increasing function on an interval \( I \). For \( c \in \mathbb{R} \) let \( A_c := g^{-1}(c, \infty) \). Case 1: \( c \notin g(I) \). If \( c \geq g(t) \) for all \( t \in I \) then \( A_c = \emptyset \). Otherwise there exist \( t \in I \) such that \( c < g(t) \). Then for \( t_c := \inf\{t | g(t) > c\} \) we have \( A_c = [t_c, \infty) \cap I \) respectively \( A_c = (t_c, \infty) \cap I \) if \( f(g_c) > c \) respectively \( g(t_c) < c \) (Note that by the strictly increasing property \( t > t_c \) implies \( g(t) > g(t_c) \) and \( t < t_c \) implies \( g(t) < g(t_c) \)). Case 2: \( c \in g(I) \) then with the uniquely determined \( t_c \) such that \( g(t_c) = c \), we have \( A_c = (t_c, \infty) \cap I \). Thus \( A_c \) is a measurable set in each case, and \( g \) is measurable. If \( f \) is increasing but not necessarily strictly then the sequence of strictly increasing functions \( f_n \) defined by \( f_n(x) = f(x) + \frac{x}{n} \) is a sequence of measurable functions by the above, which pointwise converges to \( f \). Thus \( f \) is measurable by Proposition 9.

Chapter 3, 25:) We can assume \( F \neq \emptyset \) because otherwise any constant function is a continuous extension. Define an extension \( g : \mathbb{R} \to \mathbb{R} \) as described in the hint. Let \( \mathbb{R} \setminus F = \cup_k I_k \) with countably many disjoint open intervals \( I_k \). If an interval \( I_k \) is unbounded then define \( g \) by the function, which is constant with the value of \( f(x) \) in the uniquely determined endpoint. If \( x \in I_k \) for some \( k \) then \( g(x) \) is defined by an affine function (the sum of a linear function and a constant) in a neighborhood of \( x \) and thus \( g \) is continuous at \( x \). If \( x \in F \) then let \( \varepsilon > 0 \). Since \( g|F = f \) is continuous there exists \( \delta_1 > 0 \) such that \( |g(y) - g(x)| < \varepsilon \) for all \( y \in F \) with \( |x - y| < \delta_1 \). Consider some \( k \) such that \( x \in I_k \). If \( x \in I_k \) then \( x \) is left or right endpoint of the interval \( I_k \). Suppose it is the left endpoint. Then there exists some \( \delta_2 > 0 \) such that \((x, x + \delta_2) \subseteq I_k \).

respectively decreasing sequence of real numbers.)
Similarly if $x$ is a right endpoint of some interval $I_k$ then there exists some $\delta_3 > 0$ such that $(x - \delta_3, x) \subset I_k$. It follows that $x$ is not in the closure of any other interval with index $\neq k, \ell$ because the intervals are disjoint. Note that there are three possibilities depending on whether $x$ is in the closure of none, one or two intervals. If $x$ is in the closure of two intervals then we can choose $\delta_2, \delta_3 > 0$ such that $|f(y) - f(x)| < \epsilon$ for each $x \in (x - \delta_3, x + \delta_3)$ and we have $|f(y) - f(x)| < \epsilon$ for $x \in (x - \delta_3, x + \delta_3)$ with $\delta = \min(\delta_2, \delta_3)$. Note that in this case no point of $F$ different from $x$ is in $(x - \delta, x + \delta)$. The other two cases are similar using the minimum of $\delta_1$ above and one of $\delta_2, \delta_3$. Thus $f$ is a continuous function with domain $\mathbb{R}$. (Alternatively it is not hard to argue that the oscillation of the extension $g$ on a compact neighborhood of $x \in F$ is $\leq$ to the oscillation of $f$ on that neighborhood. This also easily implies the claim.)

**Chapter 4, 17:)** $f \geq 0$ is measurable because its domain is a set of measure 0. Because $f$ is non-zero only on a set of measure, namely $E$ itself, it follows from Proposition 9 that $\int_E f = 0$.

**Chapter 4, 19:)** For $\alpha \geq 0$ the function $f : [0, 1] \to \mathbb{R}$ is piecewise continuous and bounded and thus Riemann integrable. By Theorem 4 of Chapter 3 the function is integrable and $\int_0^1 f = \frac{1}{\alpha+1}$. For $\alpha < 0$ and $n \geq 1$ let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = 0$ for $x \in [0, \frac{1}{n}]$ and let $f(x) = x^{\alpha}$ for $x \in [\frac{1}{n}, 1]$. The functions $f_n$ are Riemann integrable and thus again integrable on $[0, 1]$ with Riemann integral equal to the Lebesgue integral. The sequence $(f_n)$ is an increasing sequence of nonnegative functions converging pointwise to $f$. For $\alpha < 0$, $\alpha \neq -1$ we have $\int_0^1 f_n = \frac{1}{\alpha+1}(1 - \frac{1}{n^{\alpha+1}})$. For $\alpha = -1$ we have $\int_0^1 f_n = \ln n$. So for $-1 < \alpha < 0$ we have $\int_0^1 f_n \to \frac{1}{\alpha+1}$ while for $\alpha < -1$ we have $\int_0^1 f_n \to \infty$. Thus by the monotone convergence theorem $\int_0^1 f = \frac{1}{\alpha+1}$ for $\alpha > -1$ and $\int_0^1 f = \infty$ for $\alpha \leq -1$.

**Chapter 4, 22:)** First note that by integrability of $f$ over $\mathbb{R}$: $\int_E f \leq \int_{\mathbb{R}} f < \infty$. Suppose for the sake of contradiction that $\int_E f \neq \lim_{n \to \infty} \int_E f_n$. If $\lim_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n = \infty$ we can get by monotonicity $\lim_{n \to \infty} \int_{\mathbb{R}} f_n = \infty$, contradicting $\int_E f = \lim_{n \to \infty} \int_{\mathbb{R}} f_n < \infty$. Thus we can assume $\lim_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n < \infty$. We need this below to argue that inequalities remain strict if we add the inequality $\liminf_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n \leq \limsup_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n$ respectively the Fatou inequality $\int_{\mathbb{R} \setminus E} f \leq \liminf_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n$ to a strict inequality. First assume that $\liminf_{n \to \infty} \int_E f_n < \limsup_{n \to \infty} \int_E f_n$. Because we have $\liminf_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n \leq \limsup_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n$, by additivity over domains of integration $\liminf_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n < \limsup_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n$ contradicting $\int_{\mathbb{R}} f = \lim_{n \to \infty} \int_{\mathbb{R}} f_n$. So we can assume that $\liminf_{n \to \infty} \int_E f_n = \limsup_{n \to \infty} \int_E f_n = \lim_{n \to \infty} \int_E f_n$. By Fatou’s Lemma from the assumption for contradiction that $\int_E f < \lim_{n \to \infty} \int_E f_n = \liminf_{n \to \infty} \int_E f_n$. But Fatou’s Lemma again we know that $\int_{\mathbb{R} \setminus E} f \leq \liminf_{n \to \infty} \int_{\mathbb{R} \setminus E} f_n$ and again by the additivity over domains of integration we get $\int_{\mathbb{R}} f < \liminf_{n \to \infty} \int_{\mathbb{R}} f_n \leq \lim_{n \to \infty} \int_{\mathbb{R}} f_n$ contradicting the assumption.
Chapter 4, 24:) (i) By the simple approximation theorem there exists an increasing sequence \((\varphi_n)\) of simple nonnegative functions converging to \(f\). Let \(E_n := E \cap (-n, n)\) and \(\psi_n := \varphi_n\chi_n\). Then \(\psi_n\) is an increasing sequence of simple functions of finite support converging to \(f\). Note that for each \(x \in E\) there is \(N\) such that \(\varphi_n = \psi_n\) for \(n > N\).

(ii) Let \(s := \sup \{\int_E \varphi | \varphi\) simple of finite support and \(0 \leq \varphi \leq f\}\). Then obviously \(s \leq \int_E f\) by definition. But because of the monotone convergence theorem there exists a sequence \(\psi_n\) as in (i) with \(\int_E \psi_n \to \int_E f\), which contradicts \(s < \int_E f\). Thus \(s = \int_E f\).

Chapter 4, 25:) By Fatou’s Lemma \(\int_E f \leq \liminf_{n \to \infty} \int_E f_n\). Because \(f_n \leq f\) by monotonicity \(\int_E f_n \leq \int_E f\) and thus \(\limsup_{n \to \infty} \int_E f_n \leq \int_E f\). Thus

\[
\limsup_{n \to \infty} \int_E f_n \leq \int_E f \leq \liminf_{n \to \infty} \int_E f_n
\]

and thus \(\int_E f = \lim_{n \to \infty} \int_E f_n\).

Chapter 4, 27:) Consider the sequence \(g_n := \inf_{k \geq n} f_k\) of measurable functions. Then \(0 \leq g_n \leq f_n\) and \(g_n \to \liminf_{n \to \infty} f_n\). Applying Fatou’s Lemma to \(g_n\) gives \(\int_E \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int_E g_n \leq \liminf_{n \to \infty} \int_E f_n\) because of the monotonicity of the integral and \(\lim inf\).

Chapter 4, 30:) By possibly excising a set of measure 0 we can assume that \(|f_n| \leq g\) holds on \(E\). Let \(g_n := \inf_{k \geq n} f_k \leq f_n\). Then \(g_n \to \liminf_{n \to \infty} f_n\). Note that from \(-g \leq f_n \leq g\) for all \(n\) it also follows that \(-g \leq g_n \leq g\) for all \(n\) and thus \(|g_n| \leq g\). Thus by the Lebesgue dominated convergence theorem it follows that \(\int_E \liminf_{n \to \infty} f_n = \lim_{n \to \infty} \int_E g_n = \liminf_{n \to \infty} \int_E g_n \leq \liminf_{n \to \infty} \int_E f_n\) by monotonicity of the integral. Thus

\[
\int_E \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int_E f_n \leq \limsup_{n \to \infty} \int_E f_n
\]

Similarly let \(h_n := \sup_{k \geq n} f_k \geq f_n\) such that \(h_n \to \limsup_{n \to \infty} f_n\) and note that \(|h_n| \leq g\) as above. Again by Lebesgue dominated convergence we get \(\int_E \limsup_{n \to \infty} f_n = \lim_{n \to \infty} \int_E h_n = \limsup_{n \to \infty} \int_E h_n \geq \limsup_{n \to \infty} \int_E f_n\), so the result follows. Alternatively we could apply the first part to the sequence \(-f_n\) to get the second inequality. Also, a different proof can be given by applying Fatou’s lemma to \(g + f_n\) and \(g - f_n\), thus following the idea of the proof of the dominated convergence theorem, and not using the result.

Chapter 4, 32:) As usual, by possibly excising a set of measure zero we can assume convergence on the set \(E\). Just as in the proof of the dominated convergence theorem we can also assume that all functions are finite. Note that \(g_n + f_n, g_n - f_n \geq 0\). Since \(|f_n| \leq g_n\) for all \(n\) also \(|f| \leq g\) and thus by comparison test \(\int_E f \leq \int_E |f| \leq \int_E g < \infty\). So we know \(f_n, f\) are integrable. So by Fatou’s Lemma using linearity: \(\int_E g - \int_E f = \int_E g - f \leq \liminf_{n \to \infty} g_n - f_n = \lim_{n \to \infty} \int_E g_n - \limsup_{n \to \infty} \int_E f_n = \int_E g - \limsup_{n \to \infty} \int_E g_n\) and after subtracting \(\int_E g\) on both sides and multiplication by \(-1\) we get \(\limsup_{n \to \infty} \int_E f_n \leq \int_E f\).
Similarly by applying Fatou’s Lemma to $g + f_n$ we get $\int_E f \leq \liminf_{n \to \infty} \int_E f_n$, and the result follows.

**Chapter 4, 37:** $(E_n)$ is a decreasing sequence of measurable sets and $\cap_{n \in \mathbb{N}} E_n = \emptyset$. By continuity of integration $\lim_{n \to \infty} \int_{E_n} f = \int_{\cap_{n \in \mathbb{N}} E_n} f = \emptyset \int \emptyset = 0$. Thus for each $\varepsilon > 0$ there is a natural number $N$ such that $|\int_{E_n} f| < \varepsilon$ for $n \geq N$. Note that integrability of $f$ is defined by integrability of $|f|$ so actually the stronger statement $\int_{E_n} |f| \to 0$ for $n \to \infty$ holds.

**Chapter 4, 39:** (i) Note that $f$ integrable over $E$ implies that $f$ is integrable over all measurable subsets of $E$. Let $E_0 := \emptyset$ and for $k \geq 1$ let $C_k := E_k \setminus E_{k-1}$. Then $\cup_{n \geq 0} E_n = \cup_{n \geq 1} C_n$ and by the countable additivity of integration

$$\int_{\cup C_k} f = \lim_{n \to \infty} \int_{E_n} f = \lim_{n \to \infty} \int_{E_{k-1}} f = \lim_{n \to \infty} \int_{E_k} f - \int_{E_{k-1}} f = \int_{E_k} f - \int_{E_{k-1}} f,$$

(ii) Define $D_k := E_1 \setminus E_k$ for $k \geq 1$. The sequence $(D_k)$ is ascending and we know by (i)

$$\int_{\bigcup D_k} f = \lim_{k \to \infty} \int_{D_k} f.$$

The result follows from $\cup_{k \geq 1} D_k = E_1 \setminus \cap_{k \geq 1} E_k$ and $\int_{D_k} f = \int_{E_1} f - \int_{E_k} f$, which implies $\int_{E_1} f - \int_{D_k} f = \int_{E_1 \setminus \cap_{k \geq 1} E_k} f = \lim_{n \to \infty} \int_{E_1} f - \int_{E_n} f$ and this implies the result by subtracting $\int_{E_1} f$ and multiplying by $-1$.

**Chapter 4, 48:** Since $f, g$ are measurable we know that $fg$ is measurable by Ch. 3, Theorem 6. Moreover, if $|g| \leq M$ then $\int_E |fg| \leq \int_E |f|\cdot |g| \leq M \int_E f < \infty$ by monotonicity and linearity of $\int$ for nonnegative functions. Thus $fg$ is integrable.

**Chapter 4, 49:** (i) $\implies$ (ii): If $f = 0$ a.e. and $g$ is bounded then $fg = 0$ a.e. and $fg$ is measurable so $\int fg = 0$. (ii) $\implies$ (iii): $g = \chi_A$ is bounded and measurable and so by (ii) and $\int_A \chi_A = \int_A f$ it follows that $\int_A f = 0$ for every measurable set $A$. (iii) $\implies$ (iv): $\mathcal{O}$ is measurable and thus $\int f = 0$ for each open set $\mathcal{O}$. (iv) $\implies$ (i): Suppose that $f \neq 0$ a.e. and consider $E := \{x \in \mathbb{R} | f(x) > 0\}$ and $\{x \in \mathbb{R} | f(x) < 0\}$. If both sets have measure zero then $f = 0$ a.e. Thus we can assume that $E$ has positive measure (otherwise replace $f$ by $-f$). Since $E = \cup_{n=1}^{\infty} E_n$ with $E_n := \{x \in \mathbb{R} | f(x) > \frac{1}{n}\}$ we also find some $n \geq 1$ such that $m(E_n) > 0$ because otherwise the countable union has measure 0. Now find a $G_\delta$-set $G \supseteq E_n$ such that $m(G \setminus E_n) = 0$. Then $\int_G f = \int_{E_n} f + \int_{G \setminus E_n} f = \int_{E_n} f \geq \frac{1}{m(E_n)} > 0$. Now $G = \cap \mathcal{O}_k$ for a countable descending sequence of open sets (if $G = \cap \mathcal{U}_k$ for open sets $\mathcal{U}_k$ let $\mathcal{O}_k := \cap_{j=1}^{k} \mathcal{U}_j$. Then the $\mathcal{O}_k$ are also open and form a descending sequence). If $\int \mathcal{O}_k f = 0$ for each $k$ then also by continuity of the integral $0 = \lim_{k} \int \mathcal{O}_k f = \int_G f$.
contradicting \( \int_{\mathcal{O}} f > 0 \). Thus for some \( k \) and so for some open set \( \mathcal{O} = \mathcal{O}_k \) we have \( \int_{\mathcal{O}} f \neq 0 \). This proves (iv) \( \implies \) (i) and thus the equivalence of the four statements.

**Chapter 6, 9:** \( \iff \): Suppose \( m(E) > 0 \) and let \( (I_k) \) be a covering of \( E \) by open intervals with \( \sum \ell(I_k) < \infty \). By the Borel Cantelli lemma \( x \in I_k \) for infinitely many \( k \) can be true at most on a set of measure zero. Thus there exists at least one \( x \in E \) such that \( x \in I_k \) for at most finitely many \( k \). \( \implies \): Suppose that \( m(E) = 0 \). For each natural number \( n \) find a covering of \( E \) by countably many open intervals \( \{I_n^{(k)}\}_k \) such that \( \sum_{k=1}^{\infty} \ell(I_n^{(k)}) < \frac{1}{2^n} \). Consider the collection of all those intervals \( \{I_n^{(k)}\}_{n,k} \). Then \( \sum_{n,k} \ell(I_n^{(k)}) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \), and for each \( x \) and positive integer \( n \) there exists some \( k \) such that \( x \in I_n^{(k)} \), thus \( x \in \bigcup_{n} \bigcup_{k} I_n^{(k)} \) for infinitely many \( (n,k) \). (Note that I interpreted \( \text{belong to infinitely many } I_k \)'s as belongs to \( I_k \) for infinitely many \( k \), which actually is a slightly different statement. But this is what is needed for the next problem. Note also that the intervals necessarily are not disjoint. This seems to have been a common misunderstanding in approaches to the next problem.)

**Chapter 6, 10:** Let \( I_k = (c_k, d_k) \) be the covering of \( E \) by open intervals such that \( \sum \ell(I_k) < \infty \). Note that this makes the series \( \sum \ell(I_k \cap (-\infty, x)) \) convergent because \( \ell(I_k \cap (-\infty, x)) \leq \ell(I_k) \). Let \( f_k(x) := \ell(I_k \cap (-\infty, x)) \) such that \( f = \sum f_k \). If \( y > x \) then \( f_k(y) - f_k(x) \geq \min(\ell(x, y), (x, d_k)) \geq 0 \) for all \( k \) and thus \( f(y) \geq f(x) \). This shows that \( f \) is increasing. Let \( x \in E \) and let \( \{k_1, k_2, \ldots \} \subseteq \mathbb{N} \) be the infinite set of those \( k \) for which \( x \in I_k \). Fix a number \( N \) and consider the open interval \( I_{k_1} \cap \ldots \cap I_{k_N} \) containing \( x \). Then we can find some \( t_N > 0 \) sufficiently small such that \( x + t_N \in I_k \) for \( k = 1, \ldots, N \). Thus \( f_k(x + t_N) - f_k(x) = \ell(x, x + t_N) = t_N \) for \( k = k_1, \ldots, k_N \). Because \( f_k(x + t_N) - f_k(x) \geq 0 \) for all \( k \) it follows that \( f(x + t_N) - f(x) \geq N t_N \) and thus \( Df(x) \geq N \). Because this holds for each \( N \) we have \( Df(x) = \infty \) and thus \( f \) is not differentiable at \( x \).

**Chapter 6, 12:** Let \( f = \chi_{\mathbb{Q}} \). Case 1: \( x \in \mathbb{Q} \) and thus \( f(x) = 1 \). Consider \( 0 < |t| < h \). If \( x + t \in \mathbb{Q} \) then \( f(x + t) = 1 \) and \( \frac{f(x+t)-f(x)}{t} = 0 \). If \( x \in Q \) then \( f(x + t) = 0 \) and \( \frac{f(x+t)-f(x)}{t} = -\frac{1}{t} \). Because for each \( h > 0 \) there exist both positive and negative \( t \) with \( 0 < |t| < h \) for both cases we get \( Df(x) = \infty \) and \( Df(x) = -\infty \). Case 2: \( x \notin \mathbb{Q} \) and thus \( f(x) = 0 \). This similar with \( \frac{f(x+t)-f(x)}{t} = 0 \) if \( x \notin \mathbb{Q} \) and \( \frac{f(x+t)-f(x)}{t} = \frac{1}{t} \) if \( x + t \in \mathbb{Q} \). Again because there exist for each \( h > 0 \) both positive and negative \( t \) with \( x + t \in \mathbb{Q} \) it follows \( Df(x) = \infty \) and \( Df(x) = -\infty \). Note that \( f \) is nowhere differentiable.

**Chapter 6, 15:** For \( h > 0 \) and \( 0 < |t| < h \) we get \( \frac{f(t)-f(0)}{t} = \frac{t \sin(t)}{t} = \sin(t) \). Now for \( h > 0 \) we have \( g(t) = \frac{t}{2} \) we have \( g((-h, 0) \cup (0, h)) = (-\infty, -\frac{1}{h}) \cup (\frac{1}{h}, \infty) \), which contains an interval of length \( 2\pi \) for all \( h > 0 \). Thus by the intermediate value theorem \( \sin(\frac{t}{2})|0 < |t| < h\) \( = [-1, 1] \) and thus \( Df(0) = -1 \) and \( Df(0) = 1 \).
Chapter 6, 26:) If \( f \) were of bounded variation then it would be differentiable almost everywhere by Corollary 6 in Chapter 6. But Problem 12 above shows that \( f \) is not differentiable on \((0,1)\). Alternatively it is easy to construct partitions \( P_n \) such that \( V(f, P_n) = n \) to see \( TV(f) = \infty \) to see that \( f \) is not of bounded variation.

Chapter 6, 27:) The easiest solution is just \( f = g - h \) with \( g(x) = \sin(x) + x \) and \( h(x) = x \). Then \( g \) and \( h \) are increasing because \( g'(x) = \cos(x) + 1 \geq 0 \) and \( h'(x) = 1 > 0 \) for all \( x \in [0,2\pi] \). The standard Jordan decomposition with \( g(x) = f(x) + TV(f_{[0,\pi]}x) \) and \( h(x) = TV(f_{[0,\pi]}x) \) is also easy to deduce by calculating the total variation function using additivity: \( TV(f_{[0,\pi]}x) = \sin(x) \) for \( 0 \leq x \leq \pi/2, TV(f_{[\pi/2,\pi]}x) = 1 - \sin(x) \) for \( \pi/2 \leq x \leq 3\pi/2, TV(f_{[3\pi/2,2\pi]}x) = \sin(x) + 1 \) for \( 3\pi/2 \leq x \leq 2\pi \). Note that in particular \( TV(f_{[0,\pi]}x) = 1 \) and \( TV(f_{[\pi/2,3\pi/2]}x) = 2 \). Then \( TV(f_{[0,\pi]}x) \) is \( \sin(x) \) on \([0,\pi/2], 2 - \sin(x) \) on \([\pi/2,3\pi/2] \) and \( 4 + \sin(x) \) on \([3\pi/2,2\pi] \).

Chapter 6, 38:) Claim: \( f \) is absolutely continuous on \([a,b]\) if and only if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for every countable disjoint collection \( \{(a_k,b_k)\}_{k=1}^\infty \) of open disjoint intervals of \((a,b)\), if \( \sum_{k=1}^\infty (b_k - a_k) < \delta \) then \( \sum_{k=1}^\infty |f(b_k) - f(a_k)| < \varepsilon \). Proof: \( \Longleftarrow \): Since a finite collection is also a countable collection the definition of absolutely continuous is implied. \( \Longrightarrow \): Let \( \varepsilon > 0 \) and let \( \delta > 0 \) be such that for each finite collection \( \{(c_k,d_k)\}_{k=1}^N \) of open disjoint intervals it follows that if \( \sum_{k=1}^N (d_k - c_k) < \delta \) then \( \sum_{k=1}^N |f(d_k) - f(c_k)| < \varepsilon/2 \). Now let \( \{(a_k,b_k)\}_{k=1}^\infty \) be a countably infinite collection of open disjoint intervals with \( \sum_{k=1}^\infty (b_k - a_k) < \delta \). Then for each \( n \geq 1 \) it follows that \( \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon/2 \). Because this holds for each \( n \) it follows that \( \sum_{k=1}^\infty |f(b_k) - f(a_k)| = \lim_{n \to \infty} \sum_{k=1}^n |f(b_k) - f(a_k)| \leq \varepsilon/2 < \varepsilon \).

Chapter 6, 39:) Suppose that \( f \) is increasing. Then \( f((a,b)) \subset (f(a), f(b)) \) and so \( m^*(f((a,b))) \leq f(b) - f(a) \). Moreover if \( O \) is a countable disjoint collection of open intervals \((a_k,b_k)\) then \( m^*(f(O)) \leq \sum_{k=1}^\infty m^*(f((a_k,b_k))) \leq \sum_{k=1}^\infty (f(b_k) - f(a_k)) \). Note that we can assume \( E \subseteq (a,b) \) because changing \( E \) to \( E \setminus \{a,b\} \) and correspondingly changing \( f(E) \) to \( f(E \setminus \{a,b\}) \) will not change the measure of these sets. Next suppose that \( f \) is moreover absolutely continuous. Let \( \varepsilon > 0 \) and let \( \delta > 0 \) be such that for each countable disjoint collection of open intervals \( \{(a_k,b_k)\}_{k=1}^\infty \) with \( \sum_{k=1}^\infty (b_k - a_k) < \delta \) it follows that \( \sum_{k=1}^\infty (f(b_k) - f(a_k)) < \varepsilon \). Let \( E \) be any measurable set with \( m(E) < \delta \). We know that \( E = \bigcap_{n=1}^\infty O_n \) for a descending sequence of open sets \( O_n \subseteq (a,b) \). From the continuity of measure it follows that for \( n \) sufficiently large \( m(O_n) < \delta \). Because \( O_n \) is a countable collection of open disjoint intervals it follows that \( m^*(f(E)) \leq m^*(f(O_n)) < \varepsilon \). Conversely suppose that \( f \) is increasing and continuous, and for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( m(E) < \delta \) implies \( m^*(f(E)) < \varepsilon \). Let \( \{(a_k,b_k)\}_{k=1}^n \) be a finite collection of open disjoint intervals in \((a,b)\). Then \( m^*(f((a_k,b_k))) = m((f(a_k), f(b_k))) \). We can apply the assumption to \( E = \bigcup_{k=1}^n (a_k,b_k) \) and conclude that \( \sum_{k=1}^n (f(b_k) - f(a_k)) = \sum_{k=1}^n m((f(a_k), f(b_k))) < \varepsilon \). Remark: We have to assume that \( f \) is continuous for this direction because otherwise consider the increasing function: \( f(x) = 0 \)
for \( x \in [0, \frac{1}{2}) \) and \( f(x) = 1 \) for \( x \in [\frac{1}{2}, 1] \). Then \( f(x) \) satisfies the condition on measure because \( f([0,1]) = \{0,1\} \) has measure zero. But \( f(x) \) is not continuous and thus also not absolutely continuous.

**Chapter 6, 40:** Let \( f \) be increasing and absolutely continuous. If \( E \) has measure zero then \( m(E) < \delta \) for each \( \delta > 0 \). Thus for each \( \varepsilon > 0 \) we conclude that \( m^*(f(E)) < \varepsilon \) and thus \( m^*(f(E)) = 0 \). If the Cantor-Lebesgue function \( \varphi \) were absolutely continuous then also \( \psi \) defined by \( \psi(x) := x + \varphi(x) \) because the set of absolutely continuous functions is a real vector space. But \( \psi \) maps a set of measure 0 onto a set of positive measure. This contradicts that \( \psi \) is absolutely continuous.

**Chapter 6, 52:** The product of two absolutely continuous functions \( f, g \) on \( [a, b] \) is absolutely continuous on \( [a, b] \). The proof is essentially the same as for continuity. In fact because \( f, g \) are continuous on \( [a, b] \) it follows \(|f|, |g| \leq M \) for a constant \( M > 0 \). Let \( \varepsilon > 0 \) be given. We know that there exists \( \delta > 0 \) such that if \( \{(a_k, b_k)\}_{k=1}^n \) is a finite collection of open intervals in \( (a, b) \) with \( \sum_{k=1}^n (b_k - a_k) < \delta \) then \( \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\varepsilon}{2M} \) and \( \sum_{k=1}^n |g(b_k) - g(a_k)| < \frac{\varepsilon}{2M} \). Thus by the triangle inequality \( \sum_{k=1}^n |f(b_k)g(b_k) - f(a_k)g(a_k)| \leq \sum_{k=1}^n |f(b_k) - f(a_k)||g(b_k)| + \sum_{k=1}^n |g(b_k) - g(a_k)||f(a_k)| \leq M \sum_{k=1}^n |f(b_k) - f(a_k)| + M \sum_{k=1}^n |g(b_k) - g(a_k)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \). Thus we can apply Theorem 10 to \( fg \) and use the product rule on the set of points in \( [a, b] \) where \( fg \) is differentiable to deduce: \( \int_a^b (fg)' = \int_a^b f'g + f'g' = f(b)g(b) - f(a)g(a) \), which is the claim. (Note that the proof of the product rule requires the function to be defined in a neighborhood of a point but is a pointwise statement.)

**Chapter 6, 53:** Suppose that \( f \) is absolutely continuous and Lipschitz. Let \( x \in (a, b) \) and \( h \) sufficiently small such that \( x + t \in (a, b) \) for \( 0 < |t| \leq h \). Then \( |f(x + t) - f(x)| \leq c|t| \) for \( 0 < |t| \leq h \). It follows that \( \frac{1}{h}|f(x + h) - f(x)| \leq c \) for those \( x, t \). If \( f' \) exists at \( x \) this implies \(|f'| \leq c \). Since \( f' \) exists a.e. by Lebesgue’s theorem we conclude \(|f'| \leq c \) a.e. Conversely if \(|f'| \leq c \) a.e. on \([a, b] \) then for \( x < y \) in \([a, b] \) we get \(|f(y) - f(x)| = \int_x^y f' \leq \int_x^y |f'| \leq \int_x^y c = o(y - x) \) and we conclude that \( f \) is Lipschitz with Lipschitz constant \( c \).

**Chapter 13, 8:** First we show that \( \approx \) is an equivalence relation: \( ||x - x'|| = 0 \) so \( x \approx x \) for all \( x \) and \( \approx \) is reflexive. If \( ||y - y'|| = 0 \) then also \( ||y - x'|| = 0 \) so \( x \approx y \) implies \( y \approx x \) and their relation is symmetric. If \( x \approx y \) and \( y \approx z \) then \( ||x - y|| = 0 \) and \( ||y - z|| = 0 \) and thus \( ||x - z|| \leq ||x - y|| + ||y - z|| = 0 \) thus \( x \approx z \) and \( \approx \) is transitive. Let \([x]\) be the equivalence class of \( x \) and \( X/ \approx \) be the set of equivalence classes. Then for \( x, y \in X \) and \( \alpha, \beta \in \mathbb{R} \) define \( \alpha[x] + \beta[y] := \alpha x + \beta y \) is well-defined because \( ||x - x'|| = 0 \) and \( ||y - y'|| = 0 \) implies \( ||\alpha x + \beta y - (\alpha x' + \beta y')|| \leq ||\alpha(x - x')|| + ||\beta(y - y')|| = ||\alpha|| ||x - x'|| ||\beta|| ||y - y'|| = 0 \). The operation of addition and multiplication by scalars defined in this way on \( X/ \approx \) make \( X/ \approx \) a vector space. The axioms of vector space are directly induced from the corresponding on \( X \). Now define \( ||x|| := ||x|| \). This is well-defined because \( x \approx x' \) implies \( ||x'|| = ||x - (x - x')|| \leq ||x|| + ||x - x'|| = ||x|| \) and similarly \( ||x|| = ||x' - (x' - x)|| \leq ||x'|| \) and thus \( ||x'|| = ||x|| \). Then \|.| |
defines a norm on $X/\cong$. It suffices to show that $||x|| = 0$ implies $[x] = 0$, and the properties of a norm are induced directly from the properties of the pseudonorm on $X$. But $||x|| = 0$ implies $||x|| = 0$, which implies $x \cong 0$ and thus $[x] = [0]$. This procedure applies to $X := \{ f : [a,b] \to \mathbb{R} | \int_a^b |f|^p < \infty \}$ with the pseudonorm $||f|| := (\int_a^b |f|^p)^{1/p}$. Note that for $X$ to be a vector space functions have to be real-valued. Thus we have to restrict to representatives with values in $\mathbb{R}$. Then the quotient $X/\cong$ is $L^p[a,b]$ with norm $||.||_p$. Note that functions with $\int_a^b |f|^p < \infty$ are finite a.e. and thus we are not losing any equivalence class in comparison with respect to the equivalence relation defined on $F$ in 7.1. Also note that if $\int_a^b (f-g)^p = 0$ then $f = g$ a.e., so the equivalence classes in 7.1. and above coincide.

Chapter 13, 9): (a): If $T$ is continuous at $u_0$ then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $||u - u_0|| < \delta$ implies $||T(u - u_0)|| = ||Tu - Tu_0|| < \varepsilon$. Then if $x \in X$ and $||x|| < \delta$ we have $x = (x + u_0) - u_0$ and thus with $u := x + u_0$ we have $||u - u_0|| < \delta$ and thus $||Tx|| = ||Tu - Tu_0|| < \varepsilon$. Thus $T$ is continuous at $0$. It follows from the argument in the proof of Theorem 1 that $T$ is bounded and thus by Theorem 1 that $T$ is continuous. Since $X \not= \emptyset$ continuity implies continuity at some point. (b): If $T$ is Lipschitz with constant $c$ then $T$ is bounded with $||T|| \leq c$. Thus $T$ is continuous. If $T$ is continuous then $T$ is bounded. Then $T$ is Lipschitz with constant $||T||$. (c): Let $f : \mathbb{R} \to \mathbb{R}$ be a jump function like $f(x) = 0$ for $x < 0$ and 1 for $x \geq 0$. Then $f$ is continuous at 1 but not continuous. Let $f(x) = x^2$ defined on $\mathbb{R}$. Then $|f(x) - f(y)| = |x+y||x-y|$ and for $e.g. y = 2x$ we can make $y+x = 3x$ as large as we want, i.e. for each $N$ there exist $x, y$ such that $|f(y) - f(x)| > N|y - x|$ showing that $f$ is not Lipschitz.

Chapter 13, 20): $||x||_1 = \inf\{||x - y|| | y \in Y\} \geq 0$ is a pseudonorm: If $\lambda = 0$ then $||\lambda x|| = ||0|| = \inf\{||y - y|| | y \in Y\} = 0 = |\lambda||x||_1$ because $0 \in Y$. If $\lambda \neq 0$ then $y \in Y \iff ay \in Y$. Thus $||\lambda x||_1 = \inf\{||\lambda x - y|| | y \in Y\} = \inf\{||\lambda(x - y)|| | y \in Y\} = |\lambda|\inf\{||x - y|| | y \in Y\} = |\lambda||x||_1$. Also $||x_1 + x_2||_1 = \inf\{||x_1 + x_2 - y|| | y \in Y\} = \inf\{||x_1 + y_1 - (x_2 + y_2)|| | y_1, y_2 \in Y\} \leq \inf\{||x_1 + y_1|| | y_1 \in Y\} + \inf\{||x_2 + y_2|| | y_2 \in Y\} = ||x_1||_1 + ||x_2||_1$ using that $Y = \{y_1 - y_2 | y_1, y_2 \in Y\}$, the triangle inequality for $||.||$ and properties of the infimum. Thus $||.||_1$ is a pseudonorm on $X$. Let $X/Y$ be the normed linear space defined by the equivalence relation $x_1 \cong x_2 \iff ||x_1 - x_2||_1 = 0 \iff x_1 - x_2 = y \in Y$ for some $y \in Y$. Let $U \subset X$ open. Then $\varphi(U) = \{ [x] | x \in U \} \subset X/Y$ and let $[x] \in \varphi(U)$. If $||x|| - ||[x]||_1 = ||x - x'||_1 \geq \inf\{||x - x' - y'|| | y \in Y\} < \varepsilon$ then there exists $y_n \in Y$ such that $||x - x' - y_n|| \to ||x - x'||_1$. Let $y' = \lim y_n \in Y$ be the limit, which is in $Y$ because $Y$ is closed. Note that $[x'] = [x' + y']$ such $x' + y'$ is in the $\varepsilon$-ball at $x$. If $\varepsilon > 0$ is small enough then $x + y' \in U$ since $U$ is open and thus $[x'] = [x' + y'] \in \varphi(U)$.

This shows that the $\varepsilon$-ball at $[x]$ is contained in $\varphi(U)$.

Chapter 13, 21): Note that the projection $\varphi : X \to X/Y$ is continuous because $x_n \to x$ means $||x_n - x|| \to 0$ but $||[x_n] - [x]||_1 \leq ||x_n - x|| \to 0$ and thus also converges to 0. Let $X_0$ be a complement of $Y$ in the Banach space $X$,  


i.e. $X = X_0 \oplus Y$ (uses Zorn's lemma). Then $\varphi|X_0$ is an isomorphism of normed spaces. In fact, it is an algebraic isomorphism. Moreover since $\varphi$ is continuous on $X$ its restriction to $X_0$ is continuous. Also, since $\varphi$ maps open sets to open sets, its restriction does the same, and thus $\varphi^{-1}$ is also continuous. Now let $y_n$ be a Cauchy sequence in $X/Y$. Then the sequence $\varphi^{-1}(y_n) =: x_n \in X_0$ is a Cauchy sequence in $X_0$. Note that $\varphi^{-1}$ is linear and thus we can conclude by Problem 9 (ii) that $||x_n - x_m|| \leq c||y_n - y_m||$ for some $c > 0$. The Cauchy sequence $(x_n)$ in $X_0$ converges to a limit $x \in X$ (note that we don’t whether $X_0$ is closed in $X$). Then by continuity of $\varphi$ it follows that $y_n$ converges to $\varphi(x)$ in $X/Y$. This proves that $X/Y$ is complete.

**Chapter 13, 23:** Let $\varphi$ be a normed linear space isomorphism from $X$ to $\mathbb{R}^n$ for some $n$. Then $\varphi$ maps bounded sets to bounded sets and closed sets to closed sets. Thus $C \subseteq X$ is closed and bounded if and only if $\varphi(C)$ is closed and bounded in $\mathbb{R}^n$ if and only if this set is compact in $\mathbb{R}^n$. But the continuous map $\varphi^{-1}$ maps compact sets to compact sets, and thus $C$ is compact set in $X$.

**Chapter 13, 24:** Claim: Let $Y \subseteq X$ be a closed linear subspace, $Y \neq X$. Then for each $\varepsilon > 0$ there exists a vector $x_0 \in X$, $||x_0|| = 1$ and $||x_0 - y|| > 1 - \varepsilon$ for all $y \in Y$. Proof. If $\varepsilon \geq 1$ then $1 - \varepsilon \geq 0$ and the claim holds for each vector $x_0 \in X$. So we can assume that $0 < \varepsilon < 1$. Let $x \in X \setminus Y$. Since $Y$ is closed, $d := \inf \{||x - y|| \mid y \in Y\} > 0$, because otherwise there would be a sequence $(y_n)$ in $Y$ with $||y_n - x|| \to 0$, and $x \in \bar{Y} = Y$. Thus $d < \frac{1}{1-\varepsilon}$ and there exists $y_0 \in Y$ such that $||x - y_0|| < \frac{d}{1-\varepsilon}$. Let $x_0 := \frac{x - y_0}{||x - y_0||}$, so $||x_0|| = 1$. Now let $y \in Y$ arbitrarily. Then

$$||x_0 - y|| = \left|\frac{x}{||x - y_0||} - \frac{y_0}{||x - y_0||} - y\right| = \frac{1}{||x - y_0||} ||x - (y_0 + ||x - y_0||y)||$$

$$\geq \frac{d}{||x - y_0||} > 1 - \varepsilon$$

Here we use $y_0 + ||x - y_0||y \in Y$. 