# NF is consistent

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## 1 Remarks for readers of earlier versions

8/18/2015 replaced “relevant FM construction” section with the version in the latest series, which has more generality.

6/4/2015 Stupid mistake detected in definition of coded allowable permutation and of allowable permutation. Allowable permutation cores must be bijective. This is something I know perfectly well from working with older versions; I don’t know why it slipped under the radar here. This will probably require adjustments to some proofs to make sure that substitution maps constructed in the course of various proofs have their cores extended to bijections in a way that is harmless. The definitions have been corrected; the proof adjustments are pending. I know that the arguments work correctly because I have a better version in the works in the course of developing which I found this slip, and the required proofs do work there.
3/31/2015 Clarified the exact conditions under which interpretations of TSTU_n’s are required to exhibit the behavior of not having too many urelements. Added appropriate observations at various points where the details are relevant. 4:36 pm: improved language at that point and added labels for the conditions defining a tangled web.

3/30/2015 Must acknowledge a reader noticing that I wrote typed sets backward in their definition. Definition corrected.

3/29/2015 starting an editing pass, trying to clarify language wherever possible. Also added a remark on an easier way to establish Specker’s ambiguity results.

3/28/2015 added some clarifying language to section 3 in response to questions from a reader. Continuing in 8:18 PM version.

3/4/2015 cleaning out dated remarks below and revising the general description for readers of the earlier version. This is now the main official document, so I have changed its name.

This document gives a rather different approach from that given in the previous main document.

There are no tangled webs of cardinals in evidence: this construction actually argues for consistency of NFU + “the universe is no larger than the set of sets” by working with models of TSTU rather than TST in which there are urelements, but not too many of them, using the machinery of clans and litters to arrange for the condition that there are not too many urelements, passes to NFU by Jensen’s argument, then relies on Boffa’s result that NFU + “the universe is no larger than the set of sets” interprets NF. One can probably get tangled webs of cardinals internally here, though, by considering cardinalities of types.

The notion of coding used here is less complicated to define than versions in earlier series and should have a perfectly clear motivation (which is not to say that it is easy to see that it has all the claimed properties). A code for a symmetric set is a pair whose second term is a set of codes for elements of a support and whose first term is a collection of codes for elements of each of the orbits under permutations fixing the elements of the support (plus type information); a code for an atom is the pair of a code for its parent and a small ordinal index (plus type information); these objects are introduced formally without presupposing that their semantics is known, and the objects represented by the codes are actually defined in terms of the codes. The definitions which make all this work (and the argument supporting the assertion that this works) are recursive/inductive on set-theoretical rank of codes. Details of these recursions and inductions probably need to be added in appropriate places.

The argument that the codes make up a set (which is nontrivial due to a certain apparent non-well-foundedness of the scheme of clans and litters) is phrased in terms of set theoretical rank of codes in which supports are suitably restricted (it being shown first that every code is equivalent to a code with suitably restricted support).

The definition of allowable permutation is fine-tuned to make it easier to see that these permutations act quite freely on atoms. The machinery of strong supports and substitution extensions is present and works similarly to that in
earlier versions.

All atoms are in base types in the fused type structures. This is different from the situation in earlier series. The remark I made in an earlier version that atoms do not occur as parents of atoms is incorrect: they do, and it is quite instructive to see that this happens very much in the same way as in previous series, even though the system of clans was not designed specifically to support this in this version. The underlying motivation of the construction (which in spite of any appearances to the contrary is the same) actually forces this.

The fact that this is a Frankel-Mostowski argument is less in evidence in the main argument than in earlier versions, but this is still the case (though the “atoms” in question are now elements of the base types of embedded models of $\text{TSTU}_n$‘s rather than atoms in the sense of ZFA; the argument is actually conducted in ZFC). The links between the section on the relevant FM interpretation and what happens in the later sections need to be clarified.

I am sure that there is still plenty of tweaking to do.
2 Theoretical Background

In this document, we will prove that Quine’s “New Foundations” is consistent.

Simple type theory with urelements, or TSTU, is the first order many sorted theory with sorts indexed by the natural numbers, primitive predicates of equality, membership and sethood with all well-formed atomic sentences of the forms $x^i = y^i$, $x^i \in y^{i+1}$ and $\text{set}(x^{i+1})$, axiom schemes of comprehension $(\exists A. \text{set}(A) \land (x \in A \leftrightarrow \phi))$, extensionality 

$$(\forall a b. \text{set}(a) \land \text{set}(b) \rightarrow a = b \leftrightarrow (\forall x. x \in a \leftrightarrow y \in a))$$

and sethood $x \in a \rightarrow \text{set}(a)$. There are no explicit type indices in the axiom schemes: they are asserted with all assignments of types to variables which give well-formed formulas. Objects of type 0 whose members we would never talk about are called atoms; objects of positive types which are not sets and so have no elements are called urelements.

TSTUn is the subset of TSTU whose language contains variables only of types less than $n$, with its axioms being the axioms of TSTU expressible in that language.

NFU is the first order one-sorted theory with primitive predicates of equality, membership and sethood and axiom schemes of stratified comprehension $(\exists A. \text{set}(A) \land (x \in A \leftrightarrow \phi))$, where $\phi$ must be a formula of the language of NFU obtained from a formula of the language of TSTU by dropping type distinctions between variables (without introducing identifications of variables; such formulas are said to be stratified), extensionality 

$$(\forall a b. \text{set}(a) \land \text{set}(b) \rightarrow a = b \leftrightarrow (\forall x. x \in a \leftrightarrow y \in a))$$

and sethood $x \in a \rightarrow \text{set}(a)$.

NF is the theory obtained from NFU by dropping the sethood predicate and strengthening extensionality to $(\forall a b. a = b \leftrightarrow (\forall x. x \in a \leftrightarrow y \in a))$. Equivalently, it is NFU + “everything is a set”. Of course NF is historically prior (defined by Quine in [10]) and NFU was obtained by modifying NF (by Jensen, who defined this theory and proved its consistency in [9], 1969), but NFU has hitherto been the better understood theory of the two.

If we associate with any variable $x$ in our language a variable $x^+$ of type one higher in a bijective manner, we can inductively define an operation acting on any formula $\phi$ to obtain a formula $\phi^+$ of the same logical form with types of each variable raised by one. It is clear that if we can prove $\phi$ in TSTU, we can prove $\phi^+$. The Ambiguity Scheme asserts $\phi \leftrightarrow \phi^+$ for all sentences $\phi$. It is a theorem of Specker ([16], 1962) that the existence of a model of TSTU + Ambiguity implies the existence of a model of NFU with the same theory (after dropping type distinctions between variables). Specker actually proved this for NF, but his proof technique applies to NFU as well (and was used thus by Jensen in [9]).

Maurice Boffa showed in [2] that NF is equiconsistent with NFU + “the set of sets has the same cardinality as the universe” (there are no more objects (sets and urelements) than there are sets). Our strategy for proving the consistency
of NF will in fact be to prove consistency of TSTU + Ambiguity + “the set of sets has the same cardinality as the universe” (a typed version of this says that there are as many sets in each positive type as objects in that type).

We sketch Boffa’s argument briefly (perhaps presupposing some familiarity with doing set theory in NFU). Work in NFU. Let \( \sigma \) be a bijection from the universe onto the sets. Define \( x \in_{\sigma} y \) as \( x \in \sigma(y) \). Define \( \phi^\sigma \) for any formula in equality and membership alone as resulting by replacement of \( \in \) with \( \in_{\sigma} \) everywhere in \( \phi \). Then (strong extensionality)\(^\sigma\) is readily verified, and the comprehension axiom “\( \{x \mid \phi \} \) exists\(^\sigma\)” will be witnessed by \( \sigma^{-1}(\{x \mid \phi^\sigma\}) \) for any stratified formula \( \phi \) (noting that \( \phi^\sigma \) is stratified iff \( \phi \) is stratified).
3 A relevant FM construction

The phenomena illustrated in this section happen all through the construction that follows.

We review the requirements for the Frankel-Mostowski construction (refer to [8]). Any permutation $\pi$ of the atoms in ZFA can be extended to a class permutation of the universe by the convention $\pi(A) = \pi^{-1}A$. A Frankel-Mostowski interpretation is determined by a group $G$ of permutations of the atoms (always considered to be extended to the universe as indicated) and a collection $F$ of subgroups of $G$ (a “normal filter”) satisfying the following conditions:

1. $G \in F$,
2. $H \in F \land H \subseteq K \rightarrow K \in F$,
3. $H \in F \land K \in F \rightarrow H \cap K \in F$,
4. $\pi \in G \land H \in F \rightarrow \pi H \pi^{-1} \in F$ (normality condition).

Further, the group of permutations in $G$ such that $\pi(p) = p$ should belong to $F$ for each atom $p$.

We then say that a set $A$ is $F$-symmetric iff the group of permutations in $G$ which fix $A$ belongs to $F$. The objects in the domain of the FM interpretation are the atoms and the sets which are hereditarily $F$-symmetric. The membership relation of the FM interpretation is the restriction of the membership relation of our ambient ZFA to this domain. The grand theorem which we are using but not proving asserts that this class structure satisfies ZFA as well (but generally not choice).

We fix a regular uncountable cardinal $\kappa$. We will refer to sets of cardinality less than $\kappa$ as “small” and all other sets as “large”. We suppose that $P$ is a large set (that is, $|P| \geq \kappa$) and there is a collection of atoms $\mathbb{A}$ equinumerous with $P \times \kappa$ (which are not necessarily all of the atoms present): let $f$ be a bijection from $P \times \kappa$ onto $\mathbb{A}$ and denote $f(p, \alpha)$ as $p_\alpha$.

We refer to the set $\{p_\alpha : \alpha < \kappa\}$ as litter($p$) and refer to such sets as litters. A near-litter is defined as a subset of $\mathbb{A}$ with small symmetric difference from a litter. $p$ is called the “parent” of litter($p$), and of any near-litter with small symmetric difference from litter($p$), and of any atom $p_\alpha$.

We let $G$ be the group of permutations $\pi$ of the atoms (which may include atoms not in $\mathbb{A}$) which have the property that $\pi(\text{litter}(p)) \Delta \text{litter}(\pi(p))$ is small for each $p \in P$, and which further have the property that the action of $G$ on $P$ is restricted to an unspecified $G_0$. Another way of putting this is that a permutation is in $G$ iff it moves only a small number of elements of $\mathbb{A}$ in each litter in a way which is not expected from the action of the permutation on $P$ (and actions on $P$ may be further restricted in unspecified ways). The identity is clearly such a permutation, and inverses of such permutations and compositions of such permutations are clearly such permutations. It is worth noting that near-litters are mapped to near-litters, and only a small number of
elements of any near-litter are moved in a way not deducible from the way their parent is moved.

Let \( S \) be a small set of atoms in \( A \) and near-litters, such that the intersections of distinct near-litter elements of \( S \) are large. Let \( G_S \) be the collection of permutations in \( G \) which fix each element of \( S \) (where we apply the rule \( \pi(A) = \pi^{-1}A \)) in the case of the set elements of \( S \). The filter \( F \) consists of all subgroups of \( G \) which include a \( G_S \) as a subgroup.

The only condition on \( F \) which requires much effort is the normality condition, and it does not really require much. Suppose that \( H \) is an element of \( F \), and so includes some \( G_S \). We claim that \( \pi H \pi^{-1} \) includes \( G_{\pi(S)} \). Certainly \( \pi H \pi^{-1} \) includes \( \pi G_S \pi^{-1} \). We claim that \( \pi G_S \pi^{-1} \) includes \( G_{\pi(S)} \). Suppose that \( \sigma \in G_{\pi(S)} \); we would like to show that \( \pi^{-1} \sigma \pi \in G_S \); but this is immediate. For \( x \in S \) (whether atom or near-litter), \( \pi(x) \in \pi(S) \) is fixed by \( \sigma \), so \( \pi^{-1} \sigma \pi(x) = x \) as required.

It is worth noting that we could have required that our supports consist just of atoms and litters, but the use of near-litters simplifies the proof of the normality condition. From a given support, we obtain a support consisting entirely of atoms and litters by replacing each near-litter element \( N \) with the litter \( L \) with small symmetric difference from \( N \) and in addition the atoms in \( L \Delta N \) (which are called the anomalous elements for \( N \)).

We assume additional conditions on the action of permutations in \( G \):

**Existence of a special relation on \( A \):** There is a strictly well-founded transitive relation \( \leq \) on \( P \cup A \) such that the preimage under \( \leq \) of each element \( p_\alpha \) of \( A \) is \( \{ p \} \) and for each \( p \in P \), the collection of atoms \( \leq p \) and litters \( \text{litter}(q) \) with \( q \leq p \) is a support for \( p \), which we will call its strong support. A strong support of a general object is obtained by taking a support of the object made up of atoms and litters and adding the strong support of each parent of an atom or litter in the original support.

**A condition on free action of permutations in \( G \):** For any small collection \( \Sigma_1 \) of atoms in \( A \) and litters and small collection \( \Sigma_2 \) of elements of \( A \) such that each element of \( \Sigma_1 \) has strong support not meeting \( \Sigma_2 \), any permutation of \( \Sigma_2 \) can be extended to a permutation in \( G \) fixing each element of \( \Sigma_1 \).

**Observation that these conditions are trivially realizable:** Notice that these conditions can be made to hold by making all elements of \( P \) pure sets or at any rate requiring that no element of \( P \) has an element of \( A \) in its transitive closure: in this case every element of \( P \) has empty support and the permutations in \( G \) can be taken to be those which move each litter to a near-litter with the same parent, which clearly has the second property.

We demonstrate some properties of the resulting FM interpretation.

All small subsets of the domain of the FM interpretation are sets of the FM interpretation, which support equal to the union of the supports of their elements.
Any subcollection of the domain of the FM interpretation with small symmetric difference from a set of the FM interpretation will be a set of the FM interpretation, with support equal to the union of the supports of the elements of the small symmetric difference and the support of the set from which the difference is taken.

We say that a set \( C \) is \( \kappa \)-amorphous iff for any \( B \subseteq C \), either \( B \) or \( C \setminus B \) is small. We say that the cardinal of a \( \kappa \)-amorphous set is a \( \kappa \)-amorphous cardinal.

Every litter \( \text{litter}(p) \) is a set of the FM interpretation with support its own singleton. Further, every litter is a \( \kappa \)-amorphous set in the FM interpretation.

Let \( C \) be a subset of \( \text{litter}(p) \) with strong support \( S \), such that neither \( C \) nor \( \text{litter}(p) \setminus C \) is small. We can then choose two atoms \( p_\alpha, p_\beta \) from the litter, one in \( C \) and one not in \( C \), neither belonging to the strong support \( S \). There will be a permutation in \( G \) exchanging \( p_\alpha \) and \( p_\beta \) and fixing all elements of \( S \). This permutation moves \( C \) without moving any element of its support \( S \), which is a contradiction.

Not only is every subset of a litter either small or co-small, but every subset of the collection \( \mathcal{A} \) of atoms of interest has small symmetric difference from either a small or a co-small union of litters. We prove this in stages.

Suppose that a subset \( C \) of \( \mathcal{A} \) cuts a large number of litters (that is, there is a large collection of litters \( L \) such that \( L \cap C \) and \( L \setminus C \) are both nonempty). Suppose further that \( C \) has strong support \( S \). Any litter cut by \( C \) is cut into a small part and a large part. We can choose a litter cut by \( C \) which is not in \( S \) and no element of the small part of which belongs to \( S \) (because we are only ruling out a small collection of litters); we can then choose from this litter an element of \( C \) which is not in \( S \) and a non-element of \( C \) which is not in \( S \). A permutation in \( G \) exchanging these two atoms and fixing all elements of \( S \) will exist and will move \( C \) but not any element of \( S \), which is a contradiction.

This shows that any subset of \( \mathcal{A} \) in the FM interpretation has small symmetric difference from a union of litters which is a set.

Suppose that a union of litters \( C \) is a set of the FM interpretation including a large collection of litters and excluding a large collection of litters, with a strong support \( S \). We can then choose two litters, one included in \( C \) and one not included in \( C \), neither of which is in \( S \), and choose an element from each litter which does not belong to \( S \). A permutation in \( G \) which interchanges these two atoms and which fixes each element of \( S \) will exist and will move \( C \) to a non-union of litters (so certainly move it) and will not move any element of \( S \), which is a contradiction.

This is a contradiction.

If \( B \subseteq \mathcal{A} \) is a set of the FM interpretation, we give a precise description of the union of a large collection of litters with small symmetric difference from \( B \).

We know that for all but a small collection of litters \( L \), either \( L \) is included in \( B \) or \( L \) is disjoint from \( B \). We also know that for each litter \( L \), exactly one of the sets \( L \setminus B \) and \( L \cap B \) is large. The union of litters \( C \) that we specify is the union of all litters \( L \) such that \( L \cap B \) is large. The symmetric difference of \( L \) and \( B \) is the union of the small collection of nonempty \( L \cap B \)'s for litters \( L \) not included as subsets in \( C \) and the small collection of nonempty \( L \setminus B \)'s for \( L \) included as a subset in \( C \). This is a small union of small sets, and so is small. Moreover, \( C \)
is a set in the FM interpretation, as either the collection of litters included in \( C \) or its complement is a small set of litters which can serve as support for \( C \).

We have completed the description of the subsets of \( \mathcal{A} \) in the FM interpretation, being exactly those sets with small symmetric difference from the union of a small or co-small collection of litters.

We add a remark which is useful below. Note that a large collection of atoms which is a set of the FM interpretation must have large intersection with some litter. Otherwise, it would have to have small intersection with each of a large collection of litters, and we have seen above that a set of the FM interpretation cannot cut each of a large collection of litters.

We argue that if \( B \) and \( C \) are subsets of \( \mathcal{A} \) which are of the same cardinality in the FM interpretation, \( B \Delta C \) is small. We also observe that if \( B \) and \( C \) are large sets of the FM interpretation with small symmetric difference, it is evident that they are of the same cardinality in the FM interpretation. Suppose that \( B \) and \( C \) are of the same cardinality in the FM interpretation and \( B \Delta C \) is large: this is to be witnessed by a bijection \( f \) with strong support \( S \). We may suppose without loss of generality that \( C \setminus B \) is large (the case where \( B \setminus C \) is large is handled symmetrically). The preimage of \( C \setminus B \) is large, and so has large intersection with some litter. The image of the intersection of the preimage of \( C \setminus B \) and this litter is large, and so has large intersection with some litter. So we have a large subset of a litter in the preimage of \( C \setminus B \) mapped to a large subset of a litter. Choose two elements of the large subset in the preimage, not belonging to \( S \) and not mapped to elements of \( S \). A permutation interchanging their images and fixing all elements of \( S \) will exist and will move \( f \) but not move any element of \( S \), which is a contradiction.

It follows that it is reasonable to define \( |\text{litter}(p)| \) for any \( p \in P \) as the collection of subsets of \( \mathcal{A} \) with small symmetric difference from \( \text{litter}(p) \), as this is precisely the collection of subsets of \( \mathcal{A} \) with the same cardinality as the litter (in the internal sense of the FM interpretation).

We have shown that the power set of \( \mathcal{A} \) in the FM interpretation is extremely impoverished. In particular, it certainly conveys no set theoretical information about the structure of \( P \) in the ground interpretation. We show that \( P(\mathcal{A}) \), on the other hand, contains a subset the same size as \( P(P) \) in the sense of the FM interpretation (under the further assumption that \( P \) remains a set in the FM interpretation, which is again clearly true if \( P \) is a pure set and may be true under other conditions). This is unsurprising if we consider the size of this set in the ambient ZFA, but one must note that neither \( \mathcal{A} \) nor \( P(\mathcal{A}) \) should be expected to contain a set the same size as \( P \) in the sense of the FM interpretation.

The crucial result is that the map \( (p \in P \mapsto |\text{litter}(p)|) \) is a set of the FM interpretation. To see this, observe that any pair \( (p, |\text{litter}(p)|) \) in this set is actually fixed by any \( \pi \in G \), since elements of \( P \) are fixed, and sets with small symmetric difference from elements of \( \text{litter}(p) \) are mapped exactly to sets with small symmetric difference from \( \text{litter}(p) \). This implies further that the map \( (B \subseteq P \mapsto \bigcup\{|\text{litter}(p)| : p \in B\}) \) is a set: to see that the correct invariance holds, it is useful to recall that the cardinalities of litters are pairwise
disjoint sets. The map \((B \subseteq P \mapsto \bigcup\{|\text{litter}(p)| : p \in B\})\) is the promised injection from \(\mathcal{P}(P)\) into \(\mathcal{P}^2(\mathbb{N})\). It is a set in the FM interpretation because it is invariant under all permutations in \(G\).

The reason that this construction is interesting is that it shows how to allow structure not visible to the FM interpretation but concealed in type 0 of a model of TST (visible to the ground interpretation) to unfold not in type 1 (completely nondescript here) but in type 2. This technique can be used to cause unexpected structure to unfold at any desired type level in a model of TST, as the reader may discern in the construction that follows.
4 Interleaved interpretations of TSTU$_n$’s in fused type structures

We work in the usual set theory ZFC (it should be clear that we need nothing approaching the full strength of ZFC).

We begin by giving a general description of a kind of structure from which many structures for the language of an initial segment of TSTU (simple type theory with urelements in each type) can be extracted.

Fix a limit ordinal $\lambda$ for the rest of the paper.

A finite subset $A$ of $\lambda$ with $\min(A) + |A| \geq 3$ is called a type index. For any type index $A$ with at least two elements we define $A_1$ as $A \setminus \{\min(A)\}$ (this will be a type index). For any type index $A$, we define $A_0$ as $A$, and further $A_{i+1}$ as $(A_i)_1$ where this is defined. A type index which contains 0 as an element is called a base type index. Note that a base type index will have at least three elements.

**Definition:** A fused type structure (FTS) [of order $\lambda$] is a function $\tau$ from the type indices to sets with the following properties.

1. For distinct type indices $A, B$, $\tau(A)$ and $\tau(B)$ are disjoint sets.
2. Each element of a $\tau(A)$ where $A$ is not a base type index is of the form $(S, B)$, where $B_1 = A$ and $S \subseteq \tau(B)$. There is no claim that all such pairs $(S, B)$ belong to $\tau(A)$.

We will usually consider a fixed FTS $\tau$ and refer to $\tau(A)$ as “type $A$". Elements of types whose index is a base type index will be called atoms and will not be viewed as having extensions in the interpretations of the language of TSTU which we will give, and types indexed by base type indices will be referred to as base types.

Notice that we have arranged for a non-base type to be a disjoint union of subcollections of the power sets of the types whose indices extend its index by one downward. Notice that a general typed set $(S, A)$ is of type $A_1$ and $S$ has elements of type $A$.

With each type index $A$ (usually but not necessarily a base type index) we associate an interpretation of the language of TSTU$_{|A|}$ (and of course each TSTU$_n$ for $n < |A|$) in which each type $i$ will be realized as type $A_i$ of our fused type structure and $x^i \in y^{i+1}$ will be interpreted as $x^i \in \pi_1(y^{i+1})$. Sethood assertions $\text{set}(x^{i+1})$ will be interpreted as $\pi_2(x) = A_i$. Objects $(S, B) \in A_{i+1}$ where $B \neq A_i$ are interpreted as urelements of the positive type $i + 1$ (we reserve the term “atom” in this context for elements of type $A$ implementing type 0 objects). Of course our intention is eventually that these will give not just structures for the language of TSTU$_n$’s but actual interpretations of the theories TSTU$_n$.

For each type index $A$ (again, often but not necessarily a base type index) we can associate a partial map $\rho_A$ sending certain elements of each $\tau(A_i)$ to elements of $\mathcal{P}^i(\tau(A))$: $\rho_A$ is the identity on $\tau(A)$ and maps each object $(S, A_i)$
in each \( \tau(A_{i+1}) \) to \( \rho_A^{-1}S \). \( \rho_A \) is defined exactly at the atoms and pure sets of the interpretation of \( \text{TSTU}_{[A]} \) described in the previous paragraph (undefined at the urelements or at any set with an urelement among its iterated elements). We can say that an element \( X \) of some \( P^*(\tau(A)) \) is implemented in type \( A_i \) if there is an element \( Y \) of type \( A_i \) with \( \rho_A(Y) = X \). In this case we say that \( Y \) implements \( X \) in type \( A_i \).

We now describe additional conditions on an FTS which, if they can be met, can be used to support a proof that NF is consistent.

**Definition:** We say that an FTS \( \tau \) is “tangled” if the following three conditions hold:

- **comprehension condition:** For each type index \( A \), the axioms of \( \text{TSTU}_{[A]} \) are true in the interpretation of \( \text{TSTU}_{[A]} \) in \( \tau \) described above.
- **elementarity condition:** For each type index \( A \) and \( \alpha < \lambda \) greater than any element of \( A \), the interpretation of \( \text{TSTU}_{[A]} \) in \( \tau \) with type \( A \) as type 0 and the interpretation of \( \text{TSTU}_{[A]} \) in \( \tau \) with type \( A \cup \{\alpha\} \) as type 0 have exactly the same theory. (It follows that the theory of the interpretation of \( \text{TSTU}_n \) using a type with index \( B \) with at least \( n \) elements as type 0 depends only on \( B \setminus B_n \) (the smallest \( n \) elements of \( B \)).
- **Boffa condition (there are not many urelements):** Each interpretation of a \( \text{TSTU}_{[A]} \) satisfies assertions that there are as many sets in a given type with index higher than 1 as there are elements of that given type, for each type for which this can be asserted (that is, the interpretation of \( \text{TSTU}_{[A]} \) asserts the existence of a bijection from the collection of sets in type \( i \) to the whole of type \( i \) for each \( i \) for which this can be asserted). The qualification that this be assertable for the type refers to the fact that one needs to have types above the given type in which the bijection can be represented as a set. If the usual Kuratowski ordered pair is being used in the representation of functions, the precise qualification is that in each interpreted \( \text{TSTU}_{[A]} \) over base type with index \( A \), in each type \( i \) with index between 2 and \( |A| - 3 \) inclusive, the assertion that there is a bijection between the objects of type \( i \) and the sets of type \( i \) will hold. [The upper bound \( |A| - 3 \) could be replaced with \( |A| - 1 \) if a different representation of the ordered pair were used, but this is not essential to our argument.]

**Theorem:** If there is a tangled FTS, then NF is consistent.

**Proof:** Let \( \Sigma \) be any finite collection of sentences of the language of TSTU. Choose \( n \) so that all type indices appearing in \( \Sigma \) are less than \( n \). Define a partition of \( \lambda^n \), assigning each \( A \in \lambda^n \) to a compartment determined by the truth values of the sentences in \( \Sigma \) in the interpretation of \( \text{TSTU}_n \) in the tangled FTS with type 0 interpreted as any type indexed by a
set whose smallest \( n \) elements coincide with the elements of \( A \) (all such interpretations will agree). This partition has an infinite homogeneous set \( H \). Any interpretation with base type indexed by a set \( B \) with at least \( n + 1 \) elements which is a subset of \( H \) will be ambiguous for the sentences in \( \Sigma \). It follows that ambiguity for \( \Sigma \) is consistent with \( \text{TSTU} + \) “there are as many sets as objects” in each appropriate type, so full ambiguity is consistent with this theory by compactness, so \( \text{NFU} \) is consistent with “the collection of sets is the same size as the universe” by Specker’s results, so \( \text{NF} \) is consistent by Boffa’s results.

We indicate how to modify this to give a consistency proof for \( \text{NFU} \) similar to that of Jensen in [9]. The aim is to satisfy the first and second conditions in the definition of a tangled FTS, but not the third. Stipulate that for each type index \( A \), all pairs in \( P(\tau(A)) \times \{A\} \) belong to \( \tau(A_1) \). This is enough to ensure that all structures for the language of \( \text{TSTU}_n \) embedded in the FTS satisfy the axioms of \( \text{TSTU}_n \). To satisfy condition 2, it is enough for each base type \( \tau(B) \) to be the same size as each type \( \tau(B \cup \{\alpha\}) \) where \( \alpha > \max(B) \) (having all base types be the same size would do the trick nicely): one can verify that the interpretation of type theory based on any type \( A \) has the same theory as the interpretation based on type \( A \cup \{\alpha\} \) (where \( \alpha > \max(A) \)) by exhibiting a map sending objects of each type \( B \) with \( \max(B) = \max(A) \) to objects with type \( B \cup \{\alpha\} \) the appropriate restriction of which is an isomorphism between the two interpretations. If \( B \) is a base type, use any bijection from type \( B \) onto type \( B \cup \{\alpha\} \) as the local part of the isomorphism. If \( B \) is not a base type, define the image of \( (S,C) \in \tau(B) \) as \( (S',C \cup \{\alpha\}) \), where \( S' \) is the elementwise image of \( S \) under the isomorphism (supposing that it has already been defined for all types \( C \) with \( \min(C) < \min(B) \)). This is sufficient for the argument for Con(\( \text{NFU} \)) to go through exactly as above, but without showing consistency with \( \text{NFU} \) of the claim that there are no more objects than sets.

It is possible to modify both versions of this proof to make Specker’s ambiguity results easier to establish. The trick is to add to the language an additional predicate \( \leq \) (standing for an external well-ordering of each type) which may not be used in instances of comprehension but which will satisfy assertions that there is a \( \leq \)-least \( x \) such that \( \phi \) if there is any \( x \) such that \( \phi \). This gives a definable Hilbert symbol. Carry out the argument above with the formulas in \( \Sigma \) taken from this enriched language. One gets consistency of ambiguous \( \text{TSTU} \), with or without Boffa’s sentence, with a definable Hilbert symbol usable in instance of the ambiguity scheme. It is then straightforward to construct a term model of \( \text{TSTU} \) with integer rather than natural number types in which the types are actually isomorphic and can be identified to give a model of \( \text{NFU} \) (and of course in the case of the first construction one can then use Boffa’s argument above to convert it to a model of \( \text{NF} \)).

We now describe the motivation for the construction of an FTS which follows.
This is purely a suggestion of the motivation: we do not delude ourselves that it is easy to see that this will work.

Our intention is that each base type will have the structure of the set of atoms in the FM construction described above. The elements of each base type with index $B$ will thus be represented by notations $a_\alpha$ with $\alpha$ a small ordinal and $a$ taken from a set analogous to the set $P$ in the FM construction given above which we will denote by $\Pi(B)$ and refer to as the parent set of the base type. In type $B_2$ there will be sets implementing $|\text{litter}(a)|$ for each $a \in \Pi(B)$ (as defined in the FM construction section). The object $(\text{litter}(a),B_1)$ is represented by the briefer notation $\text{litter}(a)_B$.

The elements of non-base types will be sets symmetric with respect to a suitable group of permutations acting on the base types with induced effects on the non-base types.

We will arrange for $\Pi(B)$ to be the union of all types with index $C$ such that $C_1 = B_2$. In each interpretation of a TSTU$_a$ in which type $B_2$ implements a type $i + 1$, type $i$ will be implemented by such a type $C$. We are well aware that this presents apparent problems of non-well-foundedness: for example the entire type $B_1$ inhabited by typed subsets of $B$ will be included in $\Pi(B)$. This is not paradoxical because as in section 3 above not very many subsets of $B$ are actually implemented in the FTS, and the full set theoretical structure of $\Pi(B)$ is not visible internally to the FTS below type $B_2$. But it will be technically difficult to arrange.

By analogy with results in the section on the relevant FM interpretation above, we would expect to see a symmetric injective map taking each $a \in \Pi(B)$ to $|\text{litter}(a)|_B$, a set in type $B_2$, but the type structure requires us to represent this somewhat differently. What we will be able to construct is a symmetric injection taking each object of the form $(\{a\},C)$ [$C$ ranging over all possible values] in type $B_2$ to the set $|\text{litter}(a)|_B$ in type $B_2$. The sets $|\text{litter}(a)|$ are disjoint, so in fact we can define a injection sending any $(D,C)$ of type $B_2$ to $(\bigcup_{d \in D} |\text{litter}(d)|,B_1)$, the typed set implementing the union of all litters of elements of $D$. Now observe that this map is an injection whose domain includes all of type $B_2$ and whose range is included in the collection of typed sets of type $B_2$ with type $B_1$ elements. So there are as many sets of type $i + 1$ as objects of type $i + 1$ in any interpretation which uses types $B_1$ and $B_2$ to implement the successive types $i$ and $i + 1$.

That this injection is represented by a symmetric set will of course only hold if there are enough types in the interpretation above type $B$ for the internal representation of this bijection as a set of Kuratowski ordered pairs to be present (that is, if $B_2$ has at least four elements, so $B$ has at least six elements). If a different ordered pair were used in the internal representation of bijections, this number could be reduced, but this technical issue does not need to be addressed for our argument to work.

Further, for any non-base type $C$ with $C_1 = B_2$, $B$ can be replaced by $C \cup \{0\}$ in the above development and it will go through (in fact, $\Pi(C \cup \{0\})$ is the same set as $\Pi(B)$), so this works for any successive pair of non-base types in any interpretation, which is what is needed.
There will be quite a lot to verify to see that this works out correctly.
5 Constructing our target FTS from codes

In this section we construct a tangled FTS following the motivation given at the end of the previous section. We begin by defining a system of codes intended to represent the objects of the target FTS, though their intended semantics is not presupposed.

The underlying idea of the coding can be stated briefly (this paragraph is motivational and is no part of the definition which follows). We can represent sets $A$ which are hereditarily symmetric with respect to a group of permutations $G$ in the context of the usual FM interpretations in ZFA by codes $(\Sigma, T)$, where $\Sigma$ is a support and $T$ is a set of codes for elements of $A$ including a code for at least one element of each orbit in $A$ under permutations in $G$ fixing each element of $\Sigma$; for purposes of this paragraph we regard atoms (and other support elements) as their own codes: the referent of the code $(\Sigma, T)$ is the set of all $\pi(t)$ where $t$ is the referent of an element of $T$ and $\pi$ is a permutation in $G$ such that $\pi(s) = s$ for each $s \in \Sigma$. We will not actually be working in ZFA: we will use codes of the type described for elements of non-base types in an FTS, with elements of base types playing the role of the atoms in ZFA. We will provide codes for atoms as well (meaning here elements of base types rather than atoms in the sense of ZFA, and recalling that the base types will have structure of the sort described in section 3), using information about the parent and small ordinal index of the atom coded, and it will be seen that our supports include not only atoms but also near-litters, as in the example above, which will have codes as well. The precise definition of the class of permutations we are using will be implicit in the construction and is given explicitly afterward when the target FTS has been defined.

Note that while the objects introduced in the definition which follows are called codes and the motivation is that they have reference in the way just described, we are not in fact assuming that we know any semantics for them initially.

Definition (codes): Each code belongs to a code type associated with a type index.

For any code $T$ of code type $C$ with $C_1 = B_2$, or for any small ordinal $T$, and $\alpha < \kappa$, $(T, \alpha, B, \kappa + \lambda + 1)$ is a code (conventionally written $(T, \alpha, B, 1)$). The code type of $(T, \alpha, B, 1)$ is $B$. These codes are called atom codes. The code $T$ is called the parent code of the code $(T, \alpha, B, \kappa + \lambda + 1)$. [In terms of our motivation, the referents of the atom codes are the atoms in the base types, and we have arranged for the parent set of each base type $B$ to be the union of types $C$ with $C_1 = B_2$ as described above; the additional atoms with parents the small ordinals are used for a number of special purposes; one function that they have which can be noted immediately is that they provide a basis for the recursive construction of codes in this section]. The ordinal $\alpha$ is referred to as the litter index of $(T, \alpha, B, 1)$. 
We define $\text{clan}^*(B)$ as the set of all codes $(T, \alpha, B, 1)$. Define $\text{litter}^*(T)_B$ as $\{(T, \alpha, B, 1) : \alpha < \kappa\}$ (when the elements are well-formed codes).

We define a near-litter code as a set of the form $\{(T, \alpha, B, 1) : \alpha < \kappa\} \Delta U$, where $U$ is a small subset of $\text{clan}^*(B)$. $T$ is called the parent code of the near-litter code. An element of $U$ is called an anomalous element for the near-litter code in question. The code type of this code is $B_1$.

We define a coded support set as a small set of atom codes, near-litter codes and small ordinals.

A set code of type $B$ is a tuple $(T, \Sigma, A, \kappa + \lambda + 2)$, conventionally written $(T, \Sigma, A, 2)$, where $A$ is a type index with $B = A_1$, $\Sigma$ is a coded support set (called the support component of the code) and $T$ is a collection of codes of type $A$ (called the set component of the code). The index of the code type of any element of the support component (other than a small ordinal) must have the same maximum element as $B$ (support components violating this condition would be redundant). We impose the further condition that any element of the support component of a code is also an element of the support component of every set code in the transitive closure of the set component of the code. This condition is useful for avoiding conditions analogous to bound variable capture; it may be possible to omit it. [The intention is that the extension of the set coded by $(T, \Sigma, A, 2)$ will be the collection of images of objects coded by elements of $T$ under permutations of an appropriate kind which fix all the objects coded by elements of $\Sigma$; the sets coded will be symmetric with given support.]

All codes are constructed in the ways described above.

Note that any code appearing in the transitive closure of another code has type index with the same maximum element as the type index of the code in which it appears.

We define an equivalence relation on codes (intended to implement the relation of having the same referent, but we do not presuppose the semantics), mutually recursively with a notion of action of an allowable permutation on codes. Notice that when we define equivalence of two codes, we are presuming that we know how to compute actions on codes of set-theoretical rank less than the maximum of the ranks of the two codes, and when we define actions on codes, we are presuming that we know how to compute equivalence on codes of lower set-theoretical rank.

We first define the action of a coded allowable permutation on a code. A coded allowable permutation $\pi$ is determined by a small bijective map $\pi_0$ on codes of the shape $(\alpha, T, B, \kappa + T + 1)$, conventionally written $(\alpha, T, B, 1)$ and small ordinals, which maps elements of any $\text{clan}^*(B)$ meeting its domain to elements of the same $\text{clan}^*(B)$ (it respects clans), and maps small ordinals in its domain to small ordinals in its domain, and which has the property that elements of its domain and range are either inequivalent or equal. The coded allowable permutation $\pi$ may be called the coded substitution extension of the small map $\pi_0$. Note that the restriction of
\(\pi_0\) to ordinals must have its domain equal to its range, but this does not have to be true of the restriction of \(\pi_0\) to any \(\text{clan}^*(B)\).

\[\pi(T, \alpha, B, 1) = \pi_0(T', \alpha, B, 1)\] if this is defined for some \(T'\) equivalent to \(T\), and otherwise \((\pi(T), \alpha', B, 1)\) (\(\pi\) is taken to fix a small ordinal \(T\) not in the range of \(\pi_0\)). We describe the computation of \(\alpha':\) the elements of \(\text{litter}^*(T)\) which are not equivalent to elements of the domain of \(\pi_0\) are mapped to the elements of \(\text{litter}^*(\pi(T))\) which are not equivalent to elements of the range of \(\pi_0\), with the \(\gamma\)th element of the first set in order of increasing litter index being mapped to the \(\gamma\)th element of the second set in order of increasing litter index. Both of these collections are of size \(\kappa\), so this succeeds.

\[\pi(T, \Sigma, A, 2) = (\pi^*T, \pi^*\Sigma, A, 2),\] with the added remark that if \(L\) is a near litter code we define \(\pi(L)\) as \(\pi^*L\).

When an atom code \((T, \alpha, B, 1)\) is mapped by a coded allowable permutation to \((T', \alpha', B, 1)\) with \(T'\) not equivalent to \(\pi(T)\), both atoms are called exceptions of the allowable permutation.

Note that the inverse of an action and the composition of two actions are readily shown to be actions [up to equivalence; it is equivalence classes of codes that are actually being permuted here, not codes themselves].

We now define equivalence of codes.

We say that the action of a permutation fixes a code up to equivalence when the code obtained by applying the action of the permutation is equivalent to the original code.

Atom codes \((T, \alpha, A, 1)\) and \((T', \alpha', A', 1)\) are equivalent iff \(\alpha = \alpha', A = A', \) and \(T\) is equivalent to \(T'\) (a small ordinal \(T\) is equivalent to \(T'\) iff \(T = T'\)).

Near-litter codes are equivalent iff each element of each code is equivalent to an element of the other.

We define \(u \in^*(T, \Sigma, A, 2)\) as holding iff \(u\) is of code type \(A\) and there is \(t \in T\) and \(\pi\) a coded allowable permutation such that \(\pi(s)\) is equivalent to \(s\) for each \(s \in \Sigma\) and \(\pi(t)\) is equivalent to \(u\).

Set codes \(x\) and \(y\) are then equivalent iff they are of the same code type and for any code \(z\) of set theoretical rank less than the maximum of the ranks of the set and support components of \(x\) and \(y\), \(z \in^* x\) iff \(z \in^* y\).

We can establish that if \(x\) and \(y\) are equivalent set codes then \(z \in^* x\leftrightarrow z \in^* y\) for all codes \(z\) of appropriate type, not just those of rank less than the maximum of the ranks of the set and support components of \(x\) and \(y\). Suppose that \(x\) and \(y\) are equivalent, and \(z \in^* x\). Since \(z \in^* x\), there is a coded allowable permutation \(\pi\) (the substitution extension of a small map \(\pi_0\)) and element \(t\) of the first component of \(x\) such that \(\pi(t)\) is equivalent to \(z\). Now apply a coded allowable permutation \(\sigma\) which is the substitution extension of a map sending each atom in the range of \(\pi_0\) which is of rank less than the maximum of the ranks of the components of \(x\) and \(y\) to
an atom of minimal rank (the use of small ordinals as parents gives us a large supply of atoms of minimal rank in each clan, all in fact of the same minimal rank not depending on the clan) and not perturbing any element of the support components of either $x$ or $y$ (this is easily arranged because we have a large supply of atoms of minimal rank (all the atoms with small ordinal parents) and supports are small sets). $\sigma(\pi(t))$ is of rank lower than the ranks of the components of $x$ and $y$, so it must be equivalent to a $\pi'(u)$ for $u$ belonging to the first component of $y$ and $\pi'$ fixing the support of $y$ up to equivalence, and then $\sigma^{-1}(\pi'(u))$ is seen to be equivalent to $\pi(t)$ and so to $z$, so $z \in^* y$. The argument for the converse implication is the same with the roles of $x$ and $y$ interchanged. This result allows us to see that equivalence as defined is actually an equivalence relation. Note that we do inductively assume that equivalence is an equivalence relation on codes of rank lower than the maximum of the ranks of the components of $x$ and $y$ in the course of the argument.
5.1 Control of rank of codes

Construction (extend the support of a code): To convert a code with support component $\Sigma$ to an equivalent code with support component $\Sigma'$ extending $\Sigma$, replace the support component with $\Sigma'$ and replace the set component with the set of all codes with support component extending $\Sigma'$ (or simply all atom codes, if appropriate) which are equivalent to images of elements of the original set component under actions of allowable permutations which fix the elements of $\Sigma$ up to equivalence and which are of lower rank than the maximum of the rank of the original set component and the rank of $\Sigma'$. This code will be equivalent to the original code, by an induction on rank.

Definition (strong support): The strong support of a set code is defined as the union of the support component of the code and the union of the strong support sets of the parents of the near-litter codes in the support component of the code (the strong support of an atom as a parent is its singleton, and the strong support of a small ordinal as a parent is its singleton).

Observation: A coded substitution extension of a small map will fix a set code up to equivalence if the small map fixes all atomic elements and small ordinal elements of the strong support of the set code up to equivalence and respects each near-litter code in the strong support of the set code in the sense that any image or preimage under the small map of a code belonging to such a near-litter code is equivalent to an element of the same near-litter code. Suppose otherwise. There would be a minimal rank code in the strong support not fixed up to equivalence. It could not be an atom or small ordinal. The parent of such a near-litter element would be fixed up to equivalence, because all of its strong support elements, being of lower rank, would be sent to equivalent codes, and these include a support of the parent. So an atom code would have to be mapped into or out of the minimal rank code moved, a near-litter code, by the substitution extension, and this is ruled out by the assumed conditions on the substitution extension.

Further, the image of any set code under a coded allowable permutation is determined up to equivalence by the images of the elements of its support component. This is direct: consider applying one coded permutation followed by the inverse of another, with the same action on support elements; the composition will fix all support elements.

Further in the same vein, the image of any set code under a coded allowable permutation is determined up to equivalence by the action of the permutation on atomic elements of its strong support (and small ordinal elements) and on its exceptions mapped into or out of near-litters in its strong support. Consider applying a permutation followed by the inverse of another permutation, each agreeing with the other on atomic and small
ordinal elements of the strong support and on each other’s exceptions
mapped into or out of near-litters in the strong support. The composition
will fix all atoms in the strong support and respect all near-litters in the
strong support, so fix the object in question, which establishes the point.

**Definition (downward extension):** We say that a type index \( B \) downward
extends a type index \( A \) iff all elements of \( B \setminus A \) are smaller than all elements
of \( A \) (a type index does downward extend itself).

**Observation:** The only way that a code can contain a code as an immediate
component whose type does not downward extend its own is as an element
of the support component (of a set code) or as parent (of an atom code).

**Lemma:** Each set code of type \( A \) is equivalent to a code all of whose support
elements are either small ordinals or of types properly downward extending
\( A \) (or downward extending \( A \) in case \( A \) has just one element): the support
component of this code will be the restriction of the strong support of
the original code to the small ordinals and the types properly downward
extending \( A \).

**Proof:** We begin with the case of a code the elements of whose set component
are atoms. The substitution extension of any small map which fixes each
atom code element (and small ordinal element) of the strong support and
respects each near-litter in the strong support will fix this code up to
equivalence. Use the construction given above to produce an equivalent
code whose support is the strong support of the original code; notice that
this will not raise the rank of the code. Now observe that the orbit of any
atom in the set component under substitution extensions of maps fixing
elements of the strong support and respecting the litters of elements of the
strong support is either its equivalence class, or part of a litter (anomalous
elements of near-litters in the original support being fixed), or the entire
clan minus the atoms and near-litters in the strong support (due to our
freedom to use the substitution extension of any small injective map fixing
relevant atoms and respecting relevant near-litters). This implies that we
can remove from the new support all elements except for the atoms of type
\( A \times \{0\} \) and the near-litters of type \( A \): no other element of the support
has any control over where elements of the set component can be sent by
the class of substitution extensions we are using. The types \( A \) and \( A \times \{0\} \)
properly downward extend \( A \) as desired.

We now consider the case where the elements of the set component of our
code of type \( A \) are of type \( A \cup \{\beta\} \). In case \( A \) has one element, all compo-
ment codes of the given code have type downward extending \( A \) and we are
done. We assume as an inductive hypothesis that we have established the
result for all types with minimum of their index less than the minimum
of \( A \). We extend the support of our code to its strong support, and as-
sume that all proper component set codes of the resulting code with types
\( B \) whose index minimum is less than that of \( A \) can have their support
components restricted to codes with types properly downward extending $B_1$ without changing their equivalence class. We want to show that any permutation $\pi$ fixing all elements of the support component of our main code which are small ordinals or of types properly downward extending $A_1$ up to equivalence will fix the code up to equivalence. We show this by showing that for any $x$ in the set component, we can find a permutation $\pi'$ such that $\pi(x) = \pi'(x)$ and $\pi'$ fixes all elements of the full support of the main code. Fix such an $x$. We know that $\pi$ is the substitution extension of a small map $\sigma_0$, which may be supposed to include in its domain all elements of the strong support of $x$, which includes the strong support of the main code. Let $\pi'$ be the substitution extension of the union of the restriction of $\sigma_0$ to small ordinals and types properly downward extending $A_1$ and the identity on all atomic elements of the support of the main code. We claim first that $\pi'$ fixes the main code (up to equivalence). If it did not, it would have to move some strong support element of the main code of minimal rank, which would have to be a near-litter. It would have to fix the parent of this near-litter, because it would only move the parent if it moved some element of its strong support of lower rank, which would also be in the strong support. So $\pi'$ would have to map some atom into or out of this near-litter (in fact, into or out of the litter near it), and $\pi'$ by construction has no exceptions which lie in near-litters of the support of the main code and are not fixed. We claim that $\pi'(x) = \pi(x)$. We show this by showing that $\pi^{-1} \circ \pi'$ fixes $x$. To move $x$, it must move some element of the type restricted support of $x$. It cannot move an atom or small ordinal in the support, as all such objects are sent to the same values by $\pi$ and $\pi'$. So it must move a near-litter in the support. It cannot move it by mapping an atom into or out of it, because $\pi$ and $\pi'$ again have the same values at any such exception of $\pi$. So it must move the parent, which must have a support element of type not properly downward extending $A_1$ which is moved. The only way for a term of type properly extending $A_1$ downward to depend on terms not of this type (under our inductive assumptions about support components) is via an atom of type $A_1 \cup \{0\}$ (which has parents of type $A_2 \cup \{\delta\}$), and all such atoms in the strong support of $x$ are in fact sent to the same value by $\pi$ and $\pi'$, so in fact $\pi(x) = \pi'(x)$. This shows that the permutation $\pi$ (and its inverse) send elements of the set component to other elements of the set component of our code (up to suitable equivalences), and so that the code is fixed (up to equivalence) as desired.

**Definition (nicish code):** A code $x$ of type $A$ is nicish iff it is an atom code with a nicish parent code (small ordinals are declared nicish) or it is a set code, whose strong support contains no element of type not properly downward extending $A_1$ (downward extending $A$ in case $A$ has one element), and which has the property that each set code $y$ of type $B$ which is an element of the transitive closure of $x$ is not changed in equivalence class and becomes nicish when all elements of its support component which
belong to the support component of $x$ and are of type not properly downward extending $B_1$ (downward extending $B$ in case $B$ has one element) are removed from its support component.

**Lemma:** Every code is equivalent to a nicish code.

**Proof:** Immediate from the discussion above.

**Definition:** A nice code is a nicish code with the further properties that if it is an atom code its parent is nice (we declare small ordinals to be nice) and if it is a set code and if the elements of its set component are set codes, all of their atomic strong support elements are either elements of the strong support of the main code, or elements of elements of the strong support of the main code, or atoms with small ordinal parents, or small ordinals, and further the elements of the set components are themselves nice, and that if the main code is a set code and the elements of its set component are atoms, they are all either elements of the strong support of the main code, or elements of elements of the strong support of the main code, or atoms with small ordinal parents, or small ordinals, and further all elements of the support component of a nice code are nice.

**Observation:** Every code is equivalent to a nice code.

**Proof:** Every code is equivalent to a nicish code. In the case of an atom code, we fix its parent to be nice (we can do this by a natural ind hyp). To fix a set code to be nice, we can first fix all elements of the set and support components to be nice (mod the requirement that the elements of the set component must contain support elements of the main code), then apply suitable actions of allowable permutations fixing the strong support of the main code to convert any atoms which appear as elements of the set component or strong support elements of elements of the set component and which are not elements of the strong support of the main code or elements of elements of the strong support of the main code to atoms with small ordinal parent. This will not affect nicishness.

**Analysis of ranks of nice codes:** We do some computations on rank of nice codes. We call the support of a code the code itself if it is an atom code and its support component otherwise. We claim that there is a fixed rank $\rho(\alpha)$ for each ordinal $\alpha$ such that if the minimum of the type index $A_1$ is $\alpha$, any nice code of type $A$ has rank less than the sum of the rank of its support component and $\rho(\alpha)$, and $\rho(\alpha) \geq \rho(\beta)$ for $\beta < \alpha$.

The correct value for $\rho(1)$ is a constant, as a nice code for a set of atoms has set component with rank no higher than that of its support component, which will contain no atoms other than those involved in the support, except ones with small ordinal parents which have a fixed minimal rank.

Now consider a set code of type $A$ where the minimal index in $A_1$ is $\alpha > 0$, and suppose the result already established for $\beta < \alpha$. If $A$ is a
base type, we proceed exactly as in the case of $\rho(1)$ (the rank of the code is bounded by the rank of its support plus a constant). Otherwise, we may suppose that all elements of the set component (of type $A \cup \{\beta\}$) are replaced with set codes whose ranks are bounded by the ranks of their own supports plus $\rho(\min(A))$. Elements of the supports of elements of the set component are themselves of types properly downward extending $A$, and may further depend on a small collection of lower rank support elements of similarly restricted types. All atomic elements of the strong support of an element of the set component will be either elements of the support or elements of elements of the support of the main code or atoms with small ordinal parent, by niceness. So a maximum on the rank of an element of the set component is the rank of the support of the main code plus $\rho(\min(A)) \cdot \kappa$, so $\rho(\alpha)$ is a constant plus the supremum of $\rho(\beta) \cdot \kappa$ for $\beta < \alpha$. To see this, note that the rank of a near-litter code in the support with support elements of its parent taken from the elements and elements of elements of the main support and atoms with small ordinal parent is bounded by the rank of the main support plus $\rho(\min(A))$. Now suppose that all support elements of lower rank than one we are considering are of ranks of the form (rank of main support plus $\rho(\min(A)) \cdot \gamma$) for small ordinals $\gamma$: the rank of the support element we are considering will be bounded by the supremum of a small collection of ordinals of this form plus $\rho(\min(A))$, which is itself bounded by an ordinal of this form, and of course all ordinals of this form are bounded by the rank of the support of the main code plus $\rho(\min(A)) \cdot \kappa$.

This is enough to show that there is a bound on the rank of any nice code. The rank of an atom code is at least $\kappa + \lambda$ and is larger than the rank of its parent code by a constant amount. The rank of a nice set code is bounded by the supremum of a small set of smaller ranks of nice codes plus $\rho(\alpha)$. No code of minimal rank in its equivalence class could have rank higher than $\mu^+$, if the cardinal $\mu$ exceeds $\kappa$, $\lambda$, and each $\rho(\alpha)$. This ensures that the nice codes make up a set.
5.2 Definition of the target FTS

We can now state the definition of our target FTS. The elements of each base type $B$ are the equivalence classes of nice codes of type $B$: each atom code has the set of nice codes equivalent to it as its referent. The referent of a near-litter code is the pair of the set of referents of elements of the near-litter code and the index of the code type of its elements. The referent of each set code $(T, \Sigma, A, 2)$ is the pair $(S, A)$ where $S$ is the set of all referents of codes $x$ such that $x \in^* (T, \Sigma, A, 2)$ (which will generally include codes of higher rank that that of $(T, \Sigma, A, 2)$).
6 Verification of properties of the target FTS

We now consider the symmetry of sets in our FTS.

**Definition:** If $B$ is a base type index and $C_1 = B_2$, and $a$ is an element of type $C$, and $x$ is a code for $a$ (or if $x$ is a small ordinal), define $a_\alpha$ as the referent of $(x, \alpha, B, 1)$. We call $a$ the parent of $a_\alpha$. Clearly this does not depend on the choice of the code $x$. Define $\text{litter}(a)_B$ as the set implementing $\{a_\alpha | \alpha < \kappa\}$ in type $B_1$. Call such sets litters. Call sets with small symmetric difference from litters “near-litters”; call sets with small symmetric difference from $\text{litter}(a)_B$ near-litters (of type $B_1$) with parent $a$. Call the elements of the symmetric difference of a near-litter and the litter with the same parent the anomalous elements for the near-litter. Define $|\text{litter}(a)_B|$ as the set of type $B_2$ implementing the collection of all near-litters of type $B_1$ with parent $a$. It is worth noting that the definition of a type index ensures that $|\text{litter}(a)_B|$ always exists.

**Definition:** Let an allowable permutation be a permutation $\pi$ of atoms (elements of base types of the FTS), extended to typed sets by the natural convention $\pi(S,A) = (\pi^*S, A)$, for which there is a small bijective map $\pi_0$ sending atoms to atoms and small ordinals in its domain to small ordinals in its domain and such that if $x$ is an atom, $\pi_0(x)$ is always in the same base type (=clan) as $x$, such that $\pi$ coincides with $\pi_0$ on its domain, and $\pi$ maps the atoms in each $\text{litter}(a) \setminus \text{dom}(\pi_0)$ onto $\text{litter}(\pi(a)) \setminus \text{rng}(\pi_0)$ in increasing order of index [small ordinals not in the domain of $\pi_0$ being understood to be fixed by $\pi$]. Clearly actions of allowable permutations on codes correspond exactly to allowable permutations on the FTS. If $\pi(a_\alpha) = b_\beta \neq a_\beta$, we call both $a_\alpha$ and $b_\beta$ exceptions of $\pi$. An allowable permutation has only a small collection of exceptions (though it should be noted that not all elements of the domain of $\pi_0$ are necessarily exceptions in this sense). We call $\pi$ the substitution extension of $\pi_0$.

**Definition:** A support set is defined as a small set of atoms, near-litters, and small ordinals. An element $x$ of the FTS has support $\Sigma$ iff $\Sigma$ is a support set and each allowable permutation which fixes each element of $\Sigma$ fixes $x$.

**Theorem:** Every element of the FTS has a support. Thus every element of the FTS is in an obvious sense hereditarily symmetric with respect to the allowable permutations.

**Proof:** An atom has its own singleton as a support. The referent of a set code $(T, \Sigma, A, 2)$ has the set of referents of elements of $\Sigma$ as a support.

**Theorem:** Every subset of a type in the FTS which has support in fact corresponds to an element of the FTS.

**Proof:** Let $S$ be a subset of type $A$ which has support $\Sigma$ in the obvious sense. Construct a code whose referent is $(S, A)$: the elements of its set component will be all the nice codes for elements of $S$ with support extending $\Sigma$. 
of rank no higher than the rank of \( \Sigma + \rho(\min(A_1)) \) (or atoms in \( S \) if the elements of \( S \) are atoms) and its support component will of course be \( \Sigma \).

**Theorem:** The interpretations of the language of TSTU into the FTS actually satisfy the axioms of TSTU.

**Proof:** That they satisfy weak extensionality and sethood is already obvious.

They satisfy comprehension because any instance of comprehension with parameters symmetric with respect to the allowable permutations will define a subcollection of a type which is symmetric and thus by the previous theorem define a typed set in the FTS.

**Theorem:** The interpretations of TSTU in the FTS satisfy the assertion that each type is no larger than the collection of sets in that type.

**Proof:** Let \( B \) be a base type. The map sending each \((S, B_2 \cup \{\beta\})\) for each \( \beta \) dominated by \( B \) to \((U, B_1)\) where \( U \) is the union of the (first components of) the sets \( \{\text{letter}(a)_n\} \) for each \( a \in S \) is a bijection from all of type \( B_2 \) into the typed sets in type \( B_2 \) with elements in type \( B_1 \), and witnesses the assertion that there are just as many sets as objects in type \( B_2 \) in any interpretation with \( B_2 \) and \( B_1 \) indices of successive types. This is completely general, since for any two successive non-base types \( B_2 \) and \( C \) with \( C_1 = B_2 \), we can replace type \( B_2 \) with type \( C \cup \{0\} \) and implement an analogous map from all of type \( B_2 \) to typed sets of type \( B_2 \) with type \( C \) elements. If the type of atoms used is not a type of the interpretation we may be producing a bijection between general objects in the higher type and sets of urelements in the lower type, in terms of the particular interpretation; but this suffices. This map is represented internally to the type theory by a symmetric set (if \( B_2 \) has at least four elements, so that a representation of the bijection as a set of Kuratowski pairs can appear internally to interpretations of TSTU’s).

We now give the argument that our FTS satisfies the elementary equivalence conditions of a tangled FTS.

We define a map which gives us our first approximation to an elementary embedding from type \( A \) into type \( A \cup \{\alpha\} \), where the index \( A \) has at least two elements and \( \alpha \) dominates all elements of the index \( A \). In any nice code \( x \) with type index properly downward extending \( A_1 \), replace all type indices \( B \) which appear with \( B \cup \{\alpha\} \). It is straightforward to check that this will produce a well formed and nice code \( E_\alpha(x) \).

Not all codes of type \( A \cup \{\alpha\} \) correspond to codes \( E_\alpha(x) \). The reason for this is that \( (\beta \text{ being the maximum of the index } A_1), \{\beta,0\} \) is not a type index but \( \{\alpha, \beta, 0\} \) is. We repair this by modifying the definition of our embedding. Provide a map \( f \) which sends the small ordinals bijectively to the union of the small ordinals and a small subset of type \( \{\alpha, \beta, 0\} \), moving only a small subcollection of the small ordinals. Define \( E'_\alpha(x) \) for any code \( x \) of a type downward extending \( A_1 \) as being obtained by replacing each type index \( B \) with
Version of 6/4/2015 4:40 PM allowable permutation cores must be bijective

\[ B \cup \{ \alpha \} \] in \( x \) and further replacing each atom with small ordinal parent in a type \( \{ \beta, \gamma, 0 \} \) with the atom obtained by applying \( f \) to its parent. The map \( E^I_{\alpha}(x) \) will still fail to have some codes in its range, but any small collection of codes (or even any collection of codes of size \( \kappa \)) can be arranged to be in the range of \( E^I_{\alpha}(x) \) by choice of \( f \).

The small ordinal parents have a very basic role as allowing formation of atoms of minimal rank and a second use standing in for variables in the definition of nice codes. In the elementarity argument they have a further use. We discover that when we embed type \( A \) structures into type rank \( A \cup \alpha \) structures more parents appear in types \( \{ \alpha, \beta, \gamma, 0 \} \) than appeared in type \( \{ \beta, \gamma, 0 \} \). The idea is that we have already arranged for the type \( A \) structure to contain a lot of parents that seem to come from nowhere (the small ordinal parents); however many of the new parents from \( \{ \alpha, \beta, 0 \} \) we want to introduce into a context (we never need more than a small collection of them) we can confuse with small ordinal parents. There is additional work to show firmly that indeed the small ordinal parents in these particular types are completely indistinguishable from the interlopers from type \( \{ \alpha, \beta, 0 \} \).

The parallelism of structure between \( x \) and \( E^I_{\alpha}(x) \) is evident. The crucial additional point is that there is a coded allowable permutation fixing each element of a support \( E_{\alpha} \) up to equivalence taking \( E^I_{\alpha}(x) \) to \( E^I_{\alpha}(y) \) (up to equivalence), if and only if some allowable permutation fixing each element of \( \Sigma \) takes \( x \) to \( y \) (up to equivalence). Both directions work by parallelism of structure, plus the fact that the action of an allowable permutation on the small collection of elements of \( \{ \alpha, \beta, 0 \} \) in the range of \( f \) will correspond in the other type to the action of an allowable permutation on a small collection of small ordinal parents: in both cases the possible actions are completely free. [Small ordinal support elements have been introduced because otherwise support elements of type \( \{ \alpha, \beta, 0 \} \) would create maddening complications for the parallelism of structure].

It is now clear that the maps \( E^I_{\alpha} \) send equivalent codes to equivalent codes, so we harmlessly confuse these with maps on elements of our FTS.

Now consider a sentence \( (\exists y. \phi(x, y)) \) in an interpretation of TSTU\(_n\) with types of type rank properly downward extending \( A_1 \). We argue that \( (\exists y. \phi(E^I_{\alpha}(x), y)) \) will have the same truth value in the correlated interpretation with types properly downward extending \( A_1 \cup \{ \alpha \} \). Truth of \( (\exists y. \phi(x, y)) \) implies truth of \( (\exists y. \phi(E^I_{\alpha}(x), y)) \) immediately by parallelism of structure and the result of the previous paragraph about parallelism of existence of allowable permutations. Truth of \( (\exists y. \phi(E^I_{\alpha}(x), y)) \) might be thought not to obviously imply truth of \( (\exists y. \phi(x, y)) \) because a witness \( Y \) to the first statement might not be in the range of \( E^I_{\alpha}(x) \). However, one can modify the function \( f \) to \( f' \) which includes all atoms of type \( \{ \alpha, \beta, 0 \} \) involved in codes for \( E^I_{\alpha}(x) \) and \( Y \), arranging for \( E^I_{\alpha}(x) \) to be equal to \( E^I_{\alpha}(x) \) whereupon truth of \( \phi(E^I_{\alpha}(x), Y) \) implies truth of \( \phi(x, y^*) \) for some \( y^* \) by parallelism of structure of codes and existence of allowable permutations as desired. This is the interesting case in the argument for elementarity of the embeddings \( E^I_{\alpha} \), and the only one which presents any difficulty.
To supply a final touch to the proof that the elementarity condition holds, it is necessary to observe that we have assumed above that $A$ has more than one element. If $A$ has one element, then the theory of TSTU$_1$ based on type $A$ is trivial and of course is the same as the theory of TSTU$_1$ based on type $A \cup \{\alpha\}$ where $\alpha$ dominates $A$. 
7 Conclusions to be drawn about NF

The conclusions to be drawn about NF are rather unexciting ones.

By choosing the parameter $\lambda$ to be larger (and so to have stronger partition properties) one can show the consistency of a hierarchy of extensions of NF similar to extensions of NFU known to be consistent: one can replicate Jensen’s construction of $\omega$- and $\alpha$-models of NFU to get $\omega$- and $\alpha$-models of NF (e.g., see how we proved the existence of $\alpha$-models for the mildly impredicative fragment NFI of NF in [5]). One can show the consistency of NF + Rosser’s Axiom of Counting (see [12]), Henson’s Axiom of Cantorian Sets (see [4]), or the author’s axioms of Small and Large Ordinals (see [6], [7], [14]) in basically the same way as in NFU.

It seems clear that this argument, suitably refined, shows that the consistency strength of NF is exactly the minimum possible on previous information, that of TST + Infinity, or Mac Lane set theory (Zermelo set theory with comprehension restricted to bounded formulas). Actually showing that the consistency strength is the very lowest possible might be technically tricky, of course. I have not been concerned to do this here. It is clear from what is done here that NF is much weaker than ZFC.

By choosing the parameter $\kappa$ to be large enough, one can get local versions of Choice for sets as large as desired. The minimum value $\omega_1$ for $\kappa$ already enforces Denumerable Choice (Rosser’s assumption in his book) or Dependent Choices. It is unclear whether one can get a linear order on the universe or the Prime Ideal Theorem: that would require major changes in this construction. But certainly the question of whether NF has interesting consequences for familiar mathematical structures such as the continuum is answered in the negative: set $\kappa$ large enough and what our model of NF will say about such a structure will be entirely in accordance with what our original model of ZFC said. It is worth noting that the models of NF that we obtain are not $\kappa$-complete in the sense of containing every subset of their domains of size $\kappa$; it is well-known that a model of NF cannot contain all countable subsets of its domain. But the models of TST from which its theory is constructed will be $\kappa$-complete, so combinatorial consequences of $\kappa$-completeness will hold in the model of NF (which could further be made a $\kappa$-model by taking $\lambda$ large enough and adapting Jensen’s construction of $\alpha$-models of NFU in [9]).

The consistency of NF with the existence of a linear order on the universe or the Prime Ideal theorem is not established: questions about many weak versions of Choice remain.

The question of Maurice Boffa as to whether there is an $\omega$-model of TNT (the theory of negative types, that is TST with all integers as types, proposed by Hao Wang ([17])) is settled: an $\omega$-model of NF yields an $\omega$-model of TNT instantly. This work does not answer the question, very interesting to the author, of whether there is a model of TNT in which every set is symmetric under permutations of some lower type.

The question of the possibility of cardinals of infinite Specker rank at least in TSTU is answered and can be answered in ZFA with the same tools with
not too much additional work (the cardinalities of the types will have infinite Specker rank if the minimal index of their type is infinite), and we see that the existence of such cardinals doesn’t require much consistency strength. For those not familiar with this question, the Specker tree of a cardinal is the tree with that cardinal at the top and the children of each node (a cardinal) being its preimages under \( \alpha \mapsto 2^\alpha \). It is a theorem of Forster (a corollary of a well known theorem of Sierpinski) that the Specker tree of a cardinal is well-founded (see [3], p. 48), so has an ordinal rank, which we call the Specker rank of the cardinal. NF + Rosser’s axiom of counting proves that the Specker rank of the cardinality of the universe is infinite; it was unknown until this point whether the existence of a cardinal of infinite Specker rank was consistent even with type theory.

This work does not answer the question as to whether NF proves the existence of infinitely many infinite cardinals (discussed in [3], p. 52). A model with only finitely many infinite cardinals would have to be constructed in a totally different way.

A natural general question which arises is, to what extent are all models of NF like the ones indirectly shown to exist here? Do any of the features of this construction reflect facts about the universe of NF which we have not yet proved as theorems, or are there quite different models of NF as well?
8 References and Index

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