PROVING STATEMENTS ARE EQUIVALENT TO THE EUCLIDEAN PARALLEL POSTULATE

Euclidean Parallel Postulate. *If ℓ is a line and P an external point to ℓ then there exists a unique line m such that P lies on m and $m \parallel \ell$.*

The following is a theorem in neutral geometry; it holds independantly of the Euclidean Parallel Postulate.

Alternate Interior Angle Theorem. *Suppose ℓ and ℓ' are lines and t is a transversal. If the alternate interior angles formed by ℓ and ℓ' are congruent then $\ell \parallel \ell'$.*

The following is equivalent to the Euclidean Parallel Postulate:

Converse to the Alternate Interior Angle Theorem. *Suppose ℓ and ℓ' are lines and t is a transversal. If $\ell \parallel \ell'$ then the alternate interior angles formed by ℓ and ℓ' are congruent.*

The following is equivalent to the Euclidean Parallel Postulate:

Proclus's Axiom. *If ℓ, ℓ', t are distinct lines such that $\ell \parallel \ell'$ and t intersects ℓ then t intersects ℓ'.*

(Note: This is saying that if there is a point lying on both t and ℓ and $t \neq \ell$, then t intersects ℓ'. It would be equivalent to say that if there is a point lying on both t and ℓ, then $t = \ell$ or t intersects ℓ'.)

How do we prove that something is equivalent to the Euclidean Parallel Postulate? The first step is to understand the logical structure of such a proof. Every single statement above is an *implication*. Suppose A is the statement "$P \implies Q$" and B is the statement "$X \implies Y$". What does it mean to say that A is equivalent to B? How would we structure a proof of this equivalence? What would be the hypotheses in such a proof, and what are we allowed to use within the proof? In particular, if we have previously shown that B is equivalent to B', how does that help us in proving A is equivalent to B?

To prove that A is equivalent to B, we know we need to prove $A \implies B$ and $B \implies A$. Let's first try to show $A \implies B$.

Proof. Assume A holds.
...
Therefore B holds.
We assumed A and deduced B. This shows $A \implies B$. □

We know B is the statement "$X \implies Y$." Let's add that to the proof:

Proof. Assume A holds.
...
Therefore $X \implies Y$.
Therefore B holds.
We assumed A and deduced B. This shows $A \implies B$. □

In order to prove an implication, we should assume the hypothesis and try to reach the conclusion:

Proof. Assume A holds.
Also, assume X.
...
Therefore Y.
Therefore $X \implies Y$.
Therefore B holds.
We assumed A and deduced B. This shows $A \implies B$. □

And if you like we can add a little more explanation:

Proof. Assume A holds.
Also, assume X.
...
Therefore Y.
We assumed X and deduced Y. This shows $X \implies Y$.
Therefore B holds.
We assumed A and deduced B. This shows $A \implies B$. □

What have we assumed as hypotheses? We assumed X, the *hypothesis of* B. And we assumed that A holds, in other words, we assumed $P \implies Q$. Finally, for all the statements we are concerned with at this point in Math 311, we assumed the axioms of neutral geometry.

What have we NOT assumed? We have NOT assumed P (or Q). We assumed $P \implies Q$, but we did not assume P.

To illustrate this, consider the difference between the following two statements:

(1) If n is odd, then n^2 is odd.
(2) n is odd.

If we assume the first statement is true — if we agree that n being odd implies n^2 is odd — that does not tell us that, in fact, a given integer n is odd. So, assuming $P \implies Q$ is not the same as implying P.

However, if we assume $P \implies Q$ and we can use our other assumptions (X, neutral geometry) to deduce P (somehow!) then we can use $P \implies Q$ to deduce Q.

That is the difference between assuming $P \implies Q$, and assuming P. As you can see in the proofs above, we want to assume A is true; that is, we want to assume $P \implies Q$. We do not want to assume P (or Q, or Y). We will try to use our other assumptions (X, neutral geometry) to somehow deduce P. Then we can use $P \implies Q$, and the fact we know P, to deduce Q. Then we have to use Q (and X and neutral geometry) to somehow deduce Y. So our proof might (possibly) end up looking like this:

Proof. Assume A holds.
Also, assume X.
...
Therefore P. By hypothesis, $P \implies Q$. Hence, Q.
...
Therefore Y.
We assumed X and deduced Y. This shows $X \implies Y$.

Therefore B holds.
We assumed A and deduced B. This shows $A \implies B$. □

Our challenge as proof-writers is to fill in the gap from X to P, and the gap from Q to Y. Throughout all of this, we may use the postulates and theorems of neutral geometry, and X, and (for the latter part) P and Q (after we have shown they are true).

In fact, it may happen that $P \implies Q$ gets used more than once during the proof. For example, if P and Q are statements about parallel lines, then $P \implies Q$ might be used to say something about the parallel lines ℓ and ℓ', and then later in the proof it might be used again to say something about the parallel lines ℓ and m. This isn't required, but it can happen.

Now, what about the proof $B \implies A$? In some respects this is very similar.

Proof. Assume B holds.
Also, assume P.
...
Therefore X. By hypothesis, $X \implies Y$. Hence, Y.
...
Therefore Q.
We assumed P and deduced Q. This shows $P \implies Q$.
Therefore A holds.
We assumed B and deduced A. This shows $B \implies A$. □

That is fine as far as it goes.

But suppose we have shown that B is equivalent to B'. In this proof we are assuming B is true. That tells us that B' is also true. So in our proof we can use the statement B', in addition to B and P and neutral geometry. To fill in the gap from P to X, we can use P and B and neutral geometry and B'. To fill in the gap from Y to Q, we can use P and B and neutral geometry and B' and X and Y.

So, knowing B is equivalent to B' gives us a whole other theorem we can use in our proof of $B \implies A$. This can be extremely helpful! For example, we have shown in class that the Euclidean Parallel Postulate is equivalent to the Converse to the Alternate Interior Angles Theorem, to Proclus's Axiom, and so on. So in a proof that the Euclidean Parallel Postulate implies some other statement (say, A), we can use the Converse to the Alternate Interior Angles Theorem, Proclus's Axiom, and so on.

Does knowing that B is equivalent to B' help us to prove that $A \implies B$? Well, first consider this question: In the proof of $A \implies B$, can we assume and use the statement B'? No, we can not do that; because assuming B' is equivalent to assuming B; and we want to prove B is true, so we shouldn't assume the conclusion. So, for example, while proving that Proclus's Axiom implies the Euclidean Parallel Postulate, we should not assume the Converse to the Alternate Interior Angles Theorem, because that would be equivalent to assuming what we are trying to prove. The Alternate Interior Angles Theorem, however, would be all right to use — it is a theorem in neutral geometry. So you can see that it is important to remember which one is the theorem and which one is the converse!

It seems, then, that knowing B is equivalent to B' does not give us any new theorems we can use as steps in our proof of $A \implies B$. But it is still good for something: If we can prove $A \implies B'$, and we know B' is equivalent to B, then that would be a proof that $A \implies B$.

So this is sort of an alternative: If $A \implies B$ seems too hard, then $A \implies B'$ is an alternate route to the same result. In practice this can be useful sometimes, but it's usually best to start with $A \implies B$ directly, and try to write that proof first, before considering the alternatives.

The same consideration goes for indirect proof. You may have noticed that every one of the proof-outlines above was written as a direct proof. In my opinion, it's best to start by trying to write a direct proof. If this is not possible, then we can try an indirect proof: proof by contradiction, or contrapositive. For instance, a proof might go as follows.

Proof. Assume A holds.
Assume X. And also assume that Y is false (RAA hypothesis).
. . .
Therefore P. By hypothesis, $P \implies Q$. Hence, Q.
. . .
Therefore (some contradiction, such as Q being false).
Thus we reject the RAA hypothesis and conclude that Y is true.
We assumed X and deduced Y. This shows $X \implies Y$.
Therefore B holds.
We assumed A and deduced B. This shows $A \implies B$. □

There are other possibilities, such as contrapositive. But again, I feel it's usually best to start by trying to write a direct proof, and only then try indirect proofs.