

1. Crank-Nicolson

- (a) Using MATLAB, implement the Crank-Nicolson scheme for numerically solving the 1-D heat equation

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, & a \leq x \leq b, & t \geq 0, \\ u(x, 0) &= g(x), & u(a, t) = \phi_0(t), & u(b, t) = \phi_1(t). \end{aligned}$$

Your function should take as input: functions representing the initial condition and boundary conditions, the number of points for the uniform spatial discretization, the time span to solve the problem over, and the number of time steps. It should return the approximate solution at each time step (a matrix), a vector containing all the time steps, and a vector containing the spatial discretization. A possible function declaration is

$$[u, t, x] = \text{cnhteq}(g, p1, p2, tspan, alp, N, M)$$

The function should use MATLAB's sparse matrix library. Hand-in a copy of your function and e-mail it to me.

If possible, reuse your code from problem 1 of homework 4 for the differentiation matrix of  $u_{xx}$ . You may also use any of MATLAB's built-in linear system solvers.

- (b) Test your function on 1-D heat equation

$$\begin{aligned} u_t &= u_{xx}, & 0 \leq x \leq 1, & t \geq 0, \\ u(x, 0) &= \sin \frac{\pi x}{2} + \frac{1}{2} \sin 2\pi x, & u(0, t) = 0, & u(1, t) = e^{-\pi^2 t/4}, \end{aligned}$$

which has the exact solution

$$u(x, t) = e^{-\pi^2 t/4} \sin \frac{\pi x}{2} + \frac{1}{2} e^{-4\pi^2 t} \sin 2\pi x.$$

Use MATLAB's `waterfall` plotting function to plot the Crank-Nicolson solution of the problem over the interval  $0 \leq t \leq 1$  for  $k = 1/16$  and  $h = 1/16$ . Repeat this for  $k = 1/32$  and  $h = 1/32$ . Illustrate the second order accuracy of the method?

2. To numerically solve the equation

$$\begin{aligned} u_t + u_{xxx} &= 0, & 0 \leq x \leq 1, & t \geq 0, \\ u(x, 0) &= g(x), & u(0, t) = u(1, t) \end{aligned}$$

with periodic boundary conditions, a *naive* numerical analyst proposes to use the simple scheme of forward Euler in time and centered, second order differences in space:

$$\frac{u(x, t+k) - u(x, t)}{k} + \frac{-\frac{1}{2}u(x-2h, t) + u(x-h, t) - u(x+h, t) + \frac{1}{2}u(x+2h, t)}{h^3} = 0.$$

- (a) Using some general ODE principles, explain to the numerical analyst that this scheme is unconditionally unstable, and thus should never be used.  
 (b) Propose an alternative scheme that can be used to successfully approximate this PDE.

3. The Lax-Wendroff scheme for solving the advection equation

$$u_t + cu_x = 0 \quad (c > 0)$$

can be written as

$$\frac{1}{k}(u(x, t+k) - u(x, t)) = -\frac{c}{2h}(u(x+h, t) - u(x-h, t)) + \frac{kc^2}{2h^2}(u(x+h, t) - 2u(x, t) + u(x-h, t)).$$

In class I mentioned that this scheme is stable provided  $\frac{kc}{h} \leq 1$ . Using von Neumann stability analysis, derive this stability bound on the Lax-Wendroff scheme.

4. As discussed in class, the upwind scheme for the advection equation

$$u_t + cu_x = 0 \quad (c > 0)$$

is only first order accurate in space and time (and highly diffusive). The so-called box scheme is similar to the upwind scheme, except that it is second order accurate in space and time. The main difference, however, is that it is implicit. If  $c > 0$  then the scheme is given as

$$\frac{1}{2} \left( \frac{u(x, t+k) - u(x, t)}{k} + \frac{u(x-h, t+k) - u(x-h, t)}{k} \right) = \frac{c}{2} \left( \frac{u(x-h, t+k) - u(x, t+k)}{h} + \frac{u(x-h, t) - u(x, t)}{h} \right). \quad (1)$$

One can view this as averaging the backward (upwinded) spatial derivative at time  $t+k$  and  $t$  and averaging the time derivative at  $x-h$  and  $x$ . Note that if  $c < 0$ , one would instead use averaging of the forward (downwinded) spatial derivative and time averaging about  $x$  and  $x+h$ .

- (a) Using Maple, show that the box scheme is second order accurate in space and time. The easiest way to perform this task is to do a two-dimensional Taylor series expansion of (1) in  $h$  and  $k$  using the Maple function *mtaylor*. Then use the advection equation to eliminate any  $O(h)$  or  $O(k)$  terms.
- (b) Draw the stencil for the Box scheme and then determine its stability restriction imposed by the CFL condition.
- (c) Using von Neumann stability analysis, determine the correct stability restriction on the Box scheme.