Math 465/565 **Take Home** **Due Nov. 16, 2009**
Grady Wright **Midterm Exam** **9:30am**

Administrative Details

You may only use your class notes and the course text book (Bradie).
You may also use MATLAB or a calculator to do any computations.
No teamwork.
No discussion about the problems with anyone but me (contact via email).
No looking things up on the internet or in other books.
For full credit, show all work.

Multiple Choice: 3 points each. Circle the correct answer (only one).

1. Consider the following two approximations of e^x:

$$\text{(i)}\ \ e^x \approx \sum_{k=0}^{10} \frac{x^k}{k!}, \quad \text{(ii)}\ \ e^x \approx \frac{1}{\displaystyle\sum_{k=0}^{10} (-1)^k \frac{x^k}{k!}}.$$

   If $x < 0$ then which approximation should be used?

   a. It does not matter.
   b. (i)
   c. (ii)

2. At what rate does the following sequence of numbers converge to zero:

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1/2 \\ x_2 &= 1/4 \\ x_3 &= 1/16 \\ x_4 &= 1/256 \\ x_5 &= 1/65536 \end{aligned}$$

   a. Quadratic
   b. Linear
   c. Cubic
   d. None of the above.

3. What type of machine error occurs when evaluating the function

$$f(x) = \frac{2x - \ln\left((x+1)^2\right)}{x}$$

for values of x close to 0?

a. Overflow error

b. Underflow error

c. Loss-of-significance error

d. No machine error will occur

4. Let A be a 5−by−5 symmetric matrix with eigenvalues $\lambda_1 = 10$, $\lambda_2 = -5$, $\lambda_3 = 1$, $\lambda_4 = -1/10$, and $\lambda_5 = 0$. What is the condition number of A (i.e. $\kappa(A)$).

a. 100

b. 10

c. 50

d. None of the above.

5. Suppose the $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ is solved by means of Gaussian elimination with partial pivoting and the two-norm of the residual is found to be $4 \cdot 10^{-8}$. If $\|\mathbf{b}\|_2 = 1$ and $\kappa(A) = \|A\|_2 \|A^{-1}\|_2 = 10^4$, then we know the relative error in the computed solution $\tilde{\mathbf{x}}$ satisfies the bound

a. $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \leq 4 \cdot 10^{-4}$

b. $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \leq 8 \cdot 10^{-4}$

c. $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \leq 4 \cdot 10^{-8}$

d. All of the above

6. Consider solving the linear system $A\mathbf{x} = \mathbf{b}$ (where A is non-singular) by Gauss-Seidel

$$\mathbf{x}^{(k+1)} = T_{GS}\mathbf{x}^{(k)} + \mathbf{c}_{GS} \ .$$

Then a sufficient condition for the sequence $\{\mathbf{x}^{(k)}\}$ to converge to $\mathbf{x}$ as $k \to \infty$ is

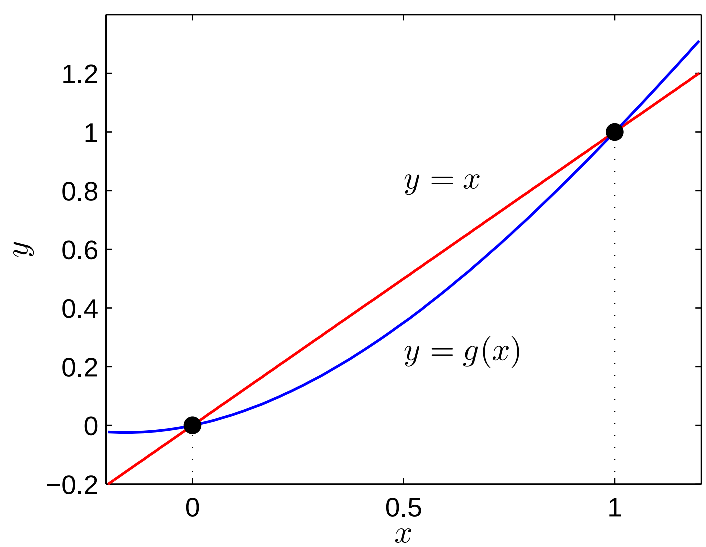
a. $\|T_{GS}\|_1 < 1$

b. $\|T_{GS}\|_\infty < 1$

c. $\rho(T_{GS}) < 1$

d. All of the above.

7. Consider the fixed point iteration for the function $g(x)$ depicted in the graph below. Based on this figure, in what interval and at what rate does it appear that the iteration will converge to the fixed point at $x = 0$.



a. $(0, 1)$, linearly

b. $(0, 0.5)$, quadratically

c. $(0, 1)$, quadratically

d. It does not converge.

8. Suppose you are given the following nodes x_j and function values $f(x_j)$ in order pairs $(x, f(x))$: $(0, 0.185)$, $(0.17, 0.214)$, $(0.34, 0.247)$, $(0.51, 0.354)$, $(0.68, 0.369)$, and $(0.85, 0.435)$. What is the best way to order this data for Neville's method if we wish to evaluate the interpolating polynomial to this data at $x = 0.41$:

   a. $(0, 0.185)$, $(0.17, 0.214)$, $(0.34, 0.247)$, $(0.51, 0.354)$, $(0.68, 0.369)$, $(0.85, 0.435)$

   b. $(0.85, 0.435)$, $(0.68, 0.369)$, $(0.51, 0.354)$, $(0.34, 0.247)$, $(0.17, 0.214)$, $(0, 0.185)$

   c. $(0, 0.185)$, $(0.85, 0.435)$, $(0.17, 0.214)$, $(0.68, 0.369)$, $(0.34, 0.247)$, $(0.51, 0.354)$

   d. $(0.34, 0.247)$, $(0.51, 0.354)$, $(0.17, 0.214)$, $(0.68, 0.369)$, $(0, 0.185)$, $(0.85, 0.435)$

9. Consider the n-by-n tridiagonal linear system of equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & -1 & 4 \end{bmatrix}.$$

   Choose the largest group of methods from the list below that can be used for computing the solution to this linear system.

   a. Crout, Cholesky, Jacobi, and Gauss-Seidel

   b. Crout, Cholesky, Jacobi, Gauss-Seidel, SOR

   c. Crout, Jacobi, and Gauss-Seidel

   d. Crout

Problems: Solve the following problems. 565 students have to answer all the questions. 465 students have to answer questions 1–4. Also, don't forget to look at the extra credit problem.

1. Linear Systems. Compute, by hand, a LU decomposition with partial pivoting of the following matrix:

$$A = \begin{bmatrix} 2 & -1 & -1 & 1 \\ 4 & -3 & -2 & 3 \\ -2 & 0 & -1 & 2 \\ 2 & -4 & -5 & 7 \end{bmatrix}$$

   You should determine, by hand, a lower triangular matrix L, an upper triangular matrix U, and a permutation matrix P such that $A = P^T LU$. Show all your steps for full credit.

2. Rootfinding. Consider using Newton's method for a cubic polynomial $p(x)$ with three distinct real zeros a, b, c. Show that if the method is started with the initial guess

$$x_0 = \frac{a+b}{2}$$

   it yields the zero c in one step.

   Hint: write $p(x)$ as the product of its zeros.

3. Fixed point iteration. For the iterations in (a)–(c) below, do the following:

   (i) Determine if the iteration is guaranteed to converge to the indicated fixed point α (provided x_0 is chosen sufficiently close to α).

(ii) If the iteration does converge, determine the order of convergence. For linear convergence, give the rate of convergence.

(iii) If the iteration does not converge, re-write the iteration in another form so that it does converge (provided x_0 is sufficiently close to α). Explain why it does converge in this new form.

(a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$ $\quad \alpha = 2$

(b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$ $\quad \alpha = 3^{1/3} \approx 1.4422$

(c) $x_{n+1} = \frac{12}{1+x_n}$ $\quad \alpha = 3$

Note: it is not sufficient to simply run the above iterations on your calculator or computer to answer (i)–(iii). You must answer these questions by applying some general theorems about fixed point iterations.

4. Interpolation.

(a) Given the following four nodes and corresponding function values

x	$f(x)$
-2	3
-1	-1
1	-1
2	5

determine the interpolationg polynomial of lowest degree possible to this data using the following methods

i. The standard Lagrange interpolation formula. You do not need to simplify your answer.

ii. The *barycentric* Lagrange interpolation formula (see problem 3 of homework 5 for the formulas). You do not need to simplify your answer, but you do need to simplify the barycentric weights.

iii. Newton's divided difference formula. You do not need to simplify your answer, but you do need to express your answer in the form it would be most computationally efficient to evaluate.

(b) Suppose you have 5 samples of the function $f(x) = 3x^4 - 2x^3 + x - 3$ at the nodes points $x_0 = 0, x_1 = h, x_2 = 3h/2, x_3 = 5h/2, x_4 = 3h$ ($h > 0$) and you construct the interpolating polynomial. Use the polynomial interpolation error formula to determine a restriction on h so that the absolute error in the interpolant is $< 10^{-10}$ for $0 \leq x \leq 3h$.

5. [565 only] Linear Systems. Suppose you have already computed the $P^T LU$ factorization of an n-by-n matrix A (i.e. $A = P^T LU$). Explain how to **efficiently** solve the n-by-n linear system

$$A^k \mathbf{x} = \mathbf{b},$$

for some positive integer k, without computing another $P^T LU$ factorization.

Note: Efficiency here means that your method should require $O(kn^2)$ operations.

6. [565 only] Interpolation. If x_j, $j = 0, 1, \ldots, n$ are distinct, we can find a unique polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

that interpolates the data points $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$. Suppose that for some reason we instead want to interpolate the same data with a unique polynomial of the form

$$p(x) = b_{n+1} x^{n+1} + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \cdots + b_1 x + b_0. \tag{1}$$

Note that $p(x)$ has no x^n term. Show that if $x_0 + x_1 + \cdots + x_n = 0$ (i.e. the sum of the interpolation nodes is zero) then there is no unique interpolating polynomial for this data.

Hint: One way to proceed is as follows:

(a) Write down the matrix that needs to be tested whether it is singular or not (this will be similar to the Vandermonde matrix).

(b) Temporarily let $x_n = x$ and note that the determinant of the matrix from the previous step then becomes a polynomial in x.

(c) Following the same argument from homework 5 problem 5, show that this polynomial of degree $n + 1$ will have roots at $x = x_0, x = x_1, \ldots, x = x_{n-1}$.

(d) Use Vieta's formula (stated below) to relate the coefficients and roots of the polynomial from the previous step to show that the remaining root has to be $x = -x_0 - x_1 - \cdots - x_{n-1}$. Hence in the original notation ($x = x_n$) the matrix is singular when $x_0 + x_1 + \cdots + x_n = 0$

One of Vieta's formulas

Suppose $p(x) = c_m x^m + c_{m-1} x^{m-1} + \cdots c_1 x + c_0$ is a polynomial with roots r_i, $i = 0, 1, \ldots, m-1$, then

$$\sum_{i=0}^{m-1} r_i = -\frac{c_{m-1}}{c_m}.$$

Extra Credit:

Prove that if $A = LD^2L^T$ for some real non-singular matrix L, and real non-singular diagonal matrix D, then A is symmetric positive definite.