Math 465/565 **Homework 7** **Due Dec. 18, 2009**
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- Instructions: 465/565 students must answer all the questions unless noted otherwise.

1. In this exercise you will be comparing trapezoidal rule to Simpson's rule for approximating the integrals:

$$I(f_1) = \int_{-1}^{1} (x-1)^2 e^{-x^2} dx = 1.87259295726583875\ldots ,$$
$$I(f_2) = \int_{-1}^{1} e^{\cos \pi x} dx = 2.53213175550401667\ldots .$$

Please answer all parts of the excersises (a)–(c) below.

(a) Using composite Trapezoidal rule, compute an approximation to $I(f_1)$ and $I(f_2)$ above for $n = 4, 8, 16, 32$. (i) Report the error in the approximations for each n in a nice table. (ii) Repeat this exercise with corrected, composite Trapezoidal rule:

$$I(f) \approx T_n(f) - \frac{h^2}{12}(f'(b) - f'(a)) .$$

(iii) Turn in a listing of your code for the corrected Trapezoidal rule. (iv) For the $I(f_1)$ results, verify that the error is decreasing like $O(h^2)$ for the standard Trapezoidal rule and $O(h^4)$ for the corrected Trapezoidal rule. (v) Does this rate of decrease in the error appear to be true for $I(f_2)$? (vi) Explain why the standard trapezoidal rule and corrected trapezoidal rule give the same results for $I(f_2)$.

(b) Using composite Simpson's rule, compute an approximation to $I(f_1)$ and $I(f_2)$ above for $n = 4, 8, 16, 32$. (i) Report the error in the approximations for each n in a nice table. (ii) Repeat this exercise with corrected, composite Simpson's rule:

$$I(f) \approx S_n(f) - \frac{h^4}{180}(f'''(b) - f'''(a)) .$$

(iii) Turn in a listing of your code for the corrected Simpson's rule. (iv) For the $I(f_1)$ results, verify that the error is decreasing like $O(h^4)$ for the standard Simpson's rule and $O(h^6)$ for the corrected Simpson's rule. (v) Does this rate of decrease in the error appear to be true for $I(f_2)$? (vi) Explain why the standard Simpson's rule and corrected Simpson's rule give the same results for $I(f_2)$. (vii) Compare the errors between the composite Trapezoidal rule and composite Simpson's rule for $I(f_2)$.

(c) Discuss the computational cost associated with each method discussed above with respect to the number of function evaluations required (not the number of additions, subtractions, multiplications, and divisions). Compare the computational cost with the resulting errors. Which method(s) appears to be the most efficient and accurate?

2. The Trapezoidal rule is based on approximating the integrand $f(x)$ with a linear polynomial in each subinterval and then integrating it exactly. If the derivative of the integrand is also available at each subinterval then we can improve upon the Trapezoidal rule by approximating the integrand with a cubic Hermite polynomial in each subinterval and integrating it exactly. This method, however, requires that $f \in C^4$ over the integration interval.

(a) Construct the cubic Hermite interpolating polynomial $p(x)$ on the interval $[x_0, x_1]$ such that

$$\begin{aligned} p(x_0) &= f(x_0) \qquad & p'(x_0) &= f'(x_0) \\ p(x_1) &= f(x_1) \qquad & p'(x_1) &= f'(x_1) \end{aligned}$$

(b) Using part a, proceed as we did in the lecture for Trapezoidal and Simpson's rule to derive the cubic Hermite quadrature formula for

$$\int_{x_0}^{x_1} f(x)dx$$

To simplify the notation, let $x_1 - x_0 = h$.

(c) Give a formula for the cubic Hermite interpolation error for part a.

(d) Again, proceeding as we did in the lecture, derive an error formula for your quadrature formula in part b.

(e) What is the degree or precision of the cubic Hermite quadrature formula?

3. This problems involves the following integral:

$$I(f_1) \quad = \quad \int_{-1}^{1} (x-1)^2 e^{-x^2} dx = 1.87259295726583875\ldots \; .$$

(a) Using Romberg integration, compute an approximation to $I(f_1)$ only for $n = 2^5$ (note that this will automatically compute the composite trapezoidal rule for $n = 4, 8, 16, 32$).

(b) Report the error in the approximations for $R_{2,2}$, $R_{3,3}$, $R_{4,4}$, $R_{5,5}$, and $R_{6,6}$ from the Romberg table and plot the these values versus the corresponding n.

(c) Turn in a listing of your code for Romberg integration.

4. [565 only] The simple correction to Trapezoidal rule $T_n(f)$ that was presented in class can be extended further to include more correction terms. The name for the correction formula is Euler-MacLaurin's formula and the general expansion is given by

$$\begin{aligned} \int_a^b f(x)dx =& T_n(f) - \frac{h^2}{12}(f'(b) - f'(a)) + \frac{h^4}{720}(f'''(b) - f'''(a)) - \\ & \frac{h^6}{30240}(f^{(5)}(b) - f^{(5)}(a)) + \cdots + \frac{B_{2m}}{(2m)!}h^{2m}(f^{(2m-1)}(b) - f^{(2m-1)}(a)) + \cdots \; , \end{aligned}$$

where B_k is the k^{th} Bernoulli number (named after Jakob Bernoulli, 1713). In addition, to providing a more accurate approximation to an integral, the Euler-MacLaurin formula can also be used to improve the convergence of a slowly converging infinite series, e.g.

$$\sum_{n=N}^{\infty} f(n) = \int_N^{\infty} f(x)dx + \frac{1}{2}f(N) - \frac{1}{12}f'(N) + \frac{1}{720}f'''(N) - \frac{1}{30240}f^{(5)}(N) + - \cdots \; . \qquad (1)$$

In this exercise, you will use Euler-MacLaurin's formula for computing Euler's constant γ, which is define by the series

$$\gamma = \lim_{n\to\infty} \left\{ \left(\sum_{k=1}^{n} \frac{1}{k} \right) - \log(n) \right\} = 0.5772156649015328\ldots \; , \qquad (2)$$

i.e the sum of the harmonic series minus the natural logarithm at infinity. Like π there are several interesting facts about γ. One of these (unlike π) is that it is not known whether γ is irrational. Fortunately, your assignment is not to prove (or disprove) that it is irrational, but to approximate its value with a more efficient method than direct summation of the slowly converging series in (2).

(a) Explain why (2) can be rewritten in the much more convenient form

$$\gamma = 1 + \sum_{k=2}^{\infty} \left[\frac{1}{k} + \log\left(1 - \frac{1}{k}\right) \right] \tag{3}$$

(b) Use Euler-MacLaurin's formula (1) on (3) to approximate γ with an accuracy $< 10^{-10}$. Your method may not sum more than 20 terms of (3) directly.

Hint: Consider splitting (3) into two parts, where the first part computes some number of terms ≤ 20 in (3) directly, and the second part applies Euler-MacLaurin's formula (1) to the remaining infinite sum.

(c) Compare your answer from part b with the result you get after directly summing the first 1000 terms of (3).