Math 465/565 **Homework 4** **Due Oct. 30, 2009**
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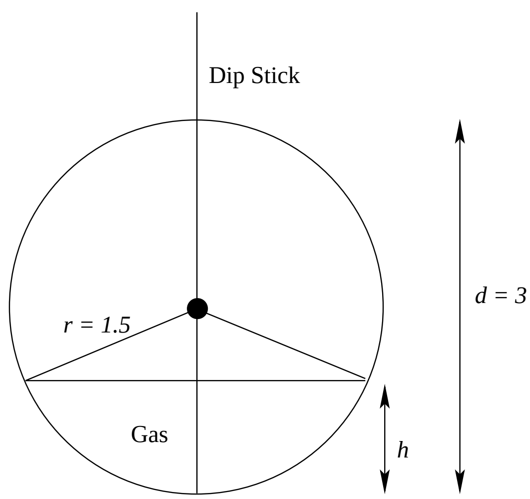
- Instructions: 465/565 students must answer all the questions unless noted otherwise.

1. Suppose you and your family are on a house boat trip in Lake Powell (`http://www.lakepowell.com`). Before you leave the dock one of the mechanics tells you that the engines on your house boat will unfortunately stop working exactly when the gas tank is 3/8 full or less. After some arguing about getting another house boat, you decide to stick with this boat and to ensure that the tanks never get below 3/8 full. Once out on the lake, you realize the gas gauge is not working so you cannot tell how much gas is in the tank. You call the mechanic on your cell phone and she tells you to simply check the level of gas manually by using a dip stick. After locating the gas tank and a dipstick, you realize there is another problem: The gas tank is perfectly cylindrical (with a diameter of 3 feet) and lies horizontal to the water. Thus, when you measure the amount of gas the length of the gas mark on the dipstick does not directly correspond to the amount of gas in the tank. For example, if the length of the mark is $3 \cdot \frac{3}{8}$, this does not mean the tank is 3/8 full. Remembering that you brought your calculator with you on your trip, determine the length of the gas mark on the dipstick that corresponds to the tank being 3/8 full. The following picture may help you get started:



You may use any rootfinding method of your choosing to solve this problem.

2. The equation
$$f(x) = x + \operatorname{erf}(2(x-1)) = x + \frac{2}{\sqrt{\pi}} \int_0^{2(x-1)} e^{-t^2} dt = 0$$
has exactly one real-valued solution. We can easily rewrite this equation as the following fixed-point equation
$$x = g(x) = -\operatorname{erf}(2(x-1)) \ . \tag{1}$$

   (a) Suppose we want to approximate the solution of (1) using the fixed-point iteration $x_{n+1} = -\operatorname{erf}(2(x_n - 1))$. Run this iteration 30 times on the computer using the initial guess $x_0 = 0$. What two values does the iteration appear to be alternating between? Are either of these values a solution of (1)?

<u>Note</u>: The error function erf(y) can be computed using the MATLAB function `erf`.

(b) Compute $\max_{-\infty<x<\infty} |g'(x)|$ and explain why the fixed-point iteration $x_{n+1} = -\text{erf}(2(x_n - 1))$ is a bad idea.

<u>Note</u>: You may use the following properties of $g(x)$:

i. $g \in C^\infty$ for $\forall x \in R$, and

ii. $g([a,b]) \subset [a,b]$ for all $a \leq -1$ and $b \geq 1$.

iii. $\lim_{x\to\infty} g(x) = -1$ and $\lim_{x\to-\infty} g(x) = 1$.

(c) Implement Steffenson's method

$$x_{n+1} = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n}$$

using, for example, MATLAB . <u>Avoid unnecessary function evaluations</u>. Use this method to find the fixed point of (1) with the initial guess $x_0 = 0$ and stopping condition $|x_{n+1} - x_n| \leq 10^{-15}$. Report the successive iterates in a 'nice' table so that you can see how convergence proceeds.

3. Jacobi and Gauss-Seidel

a. Suppose that $A \in \mathbb{R}^{n\times n}$ is strictly diagonally dominant. Prove that the Jacobi iteration matrix T_{jac} corresponding to A has the property that $\|T_{\text{jac}}\|_\infty < 1$.

b. Let the n-by-n matrix $A = D - L - U$, where L is lower triangular with zeros on its diagonal, U is upper triangular with zeros on its diagonal, and D is diagonal. If A is symmetric positive definite, show that the Gauss-Seidel iteration matrix T_{ga} corresponding to A can be written $T_{\text{ga}} = I - (D - L)^{-1}A$.

4. Successive over-relaxation (SOR(ω))

a. Consider the linear system

$$\begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 \\ -1 & -1 & 5 & -1 & -1 \\ 0 & -1 & -1 & 4 & -1 \\ 0 & 0 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \\ \\ x \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ b \\ \\ \\ \end{bmatrix}.$$

Explain why the SOR(ω) method will converge to the solution of this linear system for any initial starting guess. (Note: it is not sufficient to simply run SOR(ω) on this system to see if it converges, you must apply some theorem.)

b. Write a computer program (using MATLAB preferably) for solving a linear sytem $Ax = b$ using SOR(ω). Your program should take as input A, b, ω, and a stopping criterion ϵ. The output should be an approximate solution to the linear system. Turn in a listing of your program.

c. For $b = [3 \;\; -5 \;\; 5 \;\; -5 \;\; 3]^T$ the solution to the linear system is $x = [1 \;\; -1 \;\; 1 \;\; -1 \;\; 1]^T$. Using your SOR($\omega$) program from part b with the initial guess $x^{(0)} = 0$, solve the linear system for this b using various values of ω. Consider the system "solved" when the the approximate solution $x^{(k+1)}$ satisfies: $\left\|Ax^{(k+1)} - b\right\|_\infty < 10^{-10}$ (i.e. the residual is less than 10^{-10}). Approximate the optimal ω for this system (e.g. trial and error is sufficient). Compare the number of iterations it takes to solve the system using the optimal ω with the number it takes for various other values, including $\omega = 1$ (i.e. the Gauss-Seidel case).

5. *Rolle's theorem* can be stated as follows:

If $f(x)$ is continuous for $a \leq x \leq c$ and $f'(x)$ exists for $a < x < c$, and if $f(a) = f(c) = 0$, then there exists a number $a < \xi < c$ such that $f'(\xi) = 0$.

a. Suppose that $f(x)$ is continuous for $a \leq x \leq c$ and $f''(x)$ exists for $a < x < c$. Prove that if $f(a) = f(b) = f(c) = 0$ where $a < b < c$, then there exists a number $a < \xi < c$ such that $f''(\xi) = 0$.

b. Now suppose that $f(x)$ is continuous for $a \leq x \leq c$ and $f^{(n)}(x)$ exists for $a < x < c$. Prove that if $f(x)$ is zero at the $n+1$ distinct points $x_0, x_1, \ldots, x_n$ where $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = c$, then there exists a number $a < \xi < c$ such that $f^{(n)}(\xi) = 0$.

   Hint: Use the idea from part a and induction.

6. [565 only] Bradie, p.236, problem 14.