Math 365 **Homework 2** **Due Feb. 16, 2012**
Grady Wright

1. Loss of Significance.

   (a) Show mathematically that the functions

   $$f(x) = \cot x - 1 \qquad \text{and} \qquad g(x) = \frac{\cos 2x}{(1+\cot x)\sin^2 x}$$

   are identical.

   (b) Which function should be used when evaluating x near $\pi/4$ and $5\pi/4$? Why?

   (c) Which function should be used when evaluating x near $3\pi/4$ and $7\pi/4$? Why?

2. Matrices of size m-by-n with random entries can be created in MATLAB with the command `rand(m,n)` (note that if $m = n$ then the command `rand(n)` can be used). Create four random matrices A, B, C, and D, such that A is 2-by-3, B is 3-by-3, C is 3-by-2, and D is 3-by-3. Perform the folllowing operations on these matrices using MATLAB . If the operation cannot be performed, explain (mathematically) why not; do not just give the output from the MATLAB command window.

(a) $2A + C^T$	(b) $C - 3B$	(c) $3B - 2D$
(d) AD	(e) CA	(f) AC
(g) BD	(h) DB	(i) BC
(j) CB	(k) AB	(l) $2D^T + B$
(m) D^2	(n) A^2	(o) C^TD
(p) BA^T	(q) $-2A^T + 5C$	(r) $B^T + D$
(s) $\frac{1}{2}(B + B^T)$	(t) $\frac{1}{2}(B - B^T)$	(u) CC^T
(v) C^TC	(w) $(CC^T)^{-1}$	(x) $(C^TC)^{-1}$
(y) $B(AD)^T$	(z) ADB^T	

   Do not turn in the individual output of each of the valid operations, simply turn in the MATLAB statements you used for each of the operations.

3. NCM, problem 2.3. Please also explain where the factor $\alpha = 1/\sqrt{2}$ comes from in the set of equations.

4. NCM 2.11

5. Recall the set of equations introduced in class for modeling an n-stage countercurrent chemical extraction reactor:

   $$\begin{aligned} -(W + Sm)x_1 + Smx_2 &= -Wx_{\text{in}}, \\ Wx_{i-1} - (W + Sm)x_i + Smx_{i+1} &= 0 \qquad (i = 2, 3, 4, \ldots, n-1), \\ Wx_{n-1} - (W + Sm)x_n &= -Sy_{\text{in}}, \end{aligned}$$

   where W is the mass flow rate of the incoming water sample containing some chemical with a mass fraction x_{in}, S is the mass flow rate of the solvent containing an incoming mass fraction of the same chemical of y_{in}, and m is a constant that depends on the chemical and solvent.

   (a) Write a MATLAB function that computes the mass fraction of the chemical in the water stream at each of the n stages (i.e. x_i, $i = 1, 2, \ldots, n$). Your function should take as input W (in kg/hr), S (in kg/hr), x_{in}, y_{in}, m, and n and should output x_i, $i = 1, \ldots, n$.

It should use the `spdiags` function in matlab to form the coefficient matrix and then use the backslash operator \ to solve the system. Your function should contain no explicit looping structures. Turn-in a print out of your code.

(b) Write another MATLAB function that performs the same task as part (a), but that uses the `tridisolve` function from the NCM book to solve the linear system instead of the `spdiag` function combined with the backslash operator.

(c) Show that your functions from part (a) and (b) produce the same output by solving the n-stage countercurrent chemical extraction problem with $W = 145$ kg/hr, $S = 27$ kg/hr, $x_{\text{in}} = 0.15$, $y_{\text{in}} = 0$, $m = 7$, and $n = 70$. To compare the results of the two functions, compute the sum of the absolute value of the differences between the mass fraction values returned by both functions.

(d) Using your function from either part (a) or (b), make an *appropriate* plot of the mass fraction vs. the stages of the extraction for the problem in part (c). Here appropriate means that I can see precisely how the mass fraction decays as the water flows through the stages.

(e) By what percentage has the original chemical in the water been reduced by at the end of the extraction procedure using the parameters from part (c).

6. NCM 2.5(a)

7. As we discussed in class and in problem 2.5 from the book, if A is a symmetric positive definite matrix then A can be decomposed into the form $A = LL^T$, where L is a lower triangular matrix (or $A = R^T R$, where R is and upper triangular matrix). This decomposition is called the Cholesky decomposition.

For this problem you will use the Cholesky function built into MATLAB (`chol`), which is a very sophisticated implemenation that efficiently handles the decomposition of a *sparse* symmetric positive definite matrix by exploiting any structure formed by the non-zero entries in the matrix.

(a) Download the file "bcsstk36.mat" from the course web page and save it to your working directory in MATLAB . This file contains a sparse matrix that arises from a structural analysis of an automobile shock absorber assembly[1]. Load the matrix into matlab using the following series of commands:

```
>>load bcsstk36;
>>A = Problem.A;
```

Use the `spy` command in MATLAB to generate a plot showing the sparsity pattern of the matrix `A`. Compute the sparsity ratio of the `A` matrix by comparing the number of non-zero entries in `A` (use the `nnz` command) to the total number of entries in `A`. Report this number along with some descriptive text in the title of the `spy` plot of `A`.

(b) Compute the Cholesky decomposition of the matrix from part (a) using the "lower" option in the `chol` function (this is faster for sparse matrices than the default setting). Plot the sparsity pattern of the lower triangular matrix from the decomposition. Compute the amount of "fill-in" from the Cholesky decomposition. The "fill-in" can be defined as the ratio of the number of non-zero entries in the Cholesky decomposition to the number of non-zero entries in the original matrix. Report this number along with some descriptive text in the title of the `spy` plot of the lower triangular matrix.

(c) What you would like to happen is for the "fill-in" from the Cholesky decomposition to be as small as possible since this translate directly into a more computationally efficient method for solving the underlying linear system involving the matrix `A`. For any given sparse symmetric positive definite matrix, the "fill-in" that occurs from the Cholesky decomposition depends on how the rows and columns of the matrix are ordered. Several algorithms exist for permuting the rows and columns to try and minimize the "fill-in". All these algorithms are based on results from Graph Theory. One of the more popular (and good) of these algorithms is the Symmetric Approximate Minimum Degree Permutation

[1] see `http://www.cise.ufl.edu/research/sparse/matrices/Boeing/bcsstk36.html`

method. MATLAB contains a function for this method called `symamd`. Your goal is to use this function on the original matrix `A` from part (a) to compare the "fill-in" from the Cholesky decomposition of the permuted `A` matrix to the "fill-in" from the original one. Create plots for the sparsity pattern of the permuted `A` matrix and its Cholesky decomposition. In the latter include the "fill-in".