Final Exam Practice A: Solutions

1a) $$z = \frac{i}{\sqrt{3}+i^3} = \frac{i}{\sqrt{3}-i} = \frac{e^{i\pi/2}}{2e^{-i\pi/6}} = \frac{1}{2}e^{i[\pi/2+\pi/6]} = \boxed{\frac{1}{2}e^{i2\pi/3}}$$

b) $$z = \frac{1}{i e^{i5\pi/2}} = \frac{1}{e^{i[\pi/2+5\pi/2]}} = \frac{1}{e^{i3\pi}} = \frac{1}{-1} = \boxed{-1}$$

c) $$z^5 - 2z^3 + 4z = z(z^4 - 2z^2 + 4) = 0$$

$$\Rightarrow \underset{(1)}{z=0} \quad \& \quad \underset{(2)}{z^4 - 2z + 4 = 0}$$

(2) → Let $w = z^2 \Rightarrow w^2 - 2w + 4 = 0 \Rightarrow w = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}i$

Solutions are where

$$z=0 \quad \& \quad \underset{(3)}{z^2 - (1+\sqrt{3}i) = 0} \quad \& \quad \underset{(4)}{z^2 - (1-\sqrt{3}i) = 0}$$

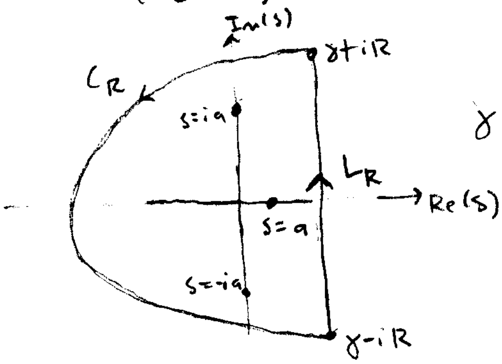
(3) → $z^2 = 1+\sqrt{3}i = 2e^{i\pi[1/3+2n]} \Rightarrow z = \sqrt{2}\,e^{i\pi[1/6+n]}$ $\quad n=0,1$

(4) → $z^2 = 1-\sqrt{3}i = 2e^{i\pi[-1/3+2n]} \Rightarrow z = \sqrt{2}\,e^{i\pi[-1/6+n]}$ $\quad n=0,1$

Roots: $$\boxed{z_0=0,\quad z_1 = \sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}},\quad z_2 = -\sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}},\quad z_3 = -\sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}},\quad z_4\ \sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}}}$$

3) $\hat{F}(s) = \dfrac{s}{(s-a)(s^2+a^2)}$ $\quad (a > 0)$ $\quad \left(\begin{array}{c}\text{Poles at} \\ s=a \ \& \ s=ia\end{array}\right)$

$$f(t) = \frac{1}{2\pi i} \text{ P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \hat{F}(s)\,ds \qquad (t > 0)$$



$\gamma > a > 0$

First, on C_R $\quad (s = \gamma + Re^{i\theta})$

$$\left|\hat{F}(s)\right| \leq \frac{|s|}{\big||s|-|a|\big|\ \big||s|^2-|a|^2\big|}$$

$$\leq \frac{\gamma + R}{|R-\gamma-a|\,|(R-\gamma)^2-a^2|} = M_R \quad \& \quad \lim_{R\to\infty} M_R = 0$$

$$\lim_{R\to\infty} \int_{C_R} e^{st}\hat{F}(s)\,ds = 0$$

$$\Rightarrow f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} e^{st}\hat{F}(s)\,ds = \sum_{j=1}^{3} \operatorname*{Res}_{s=s_j}\left[\frac{se^{st}}{(s-a)(s+ia)(s-ia)}\right]$$

where $s_1 = a$, $s_2 = ia$, $s_3 = -ia$

$$= \frac{ae^{at}}{(a+ia)(a-ia)} + \frac{iae^{iat}}{(ia-a)(2ia)} + \frac{-iae^{-iat}}{(-ia-a)(-2ia)}$$

$$= \frac{e^{at}}{2a} + \frac{ae^{iat}}{2ia(a+ia)} - \frac{ae^{-iat}}{2ia(a-ia)}$$

$$= \frac{e^{at}}{2a} + \frac{1}{2\sqrt{2}ia}\Big[\underbrace{\underbrace{e^{iat}e^{-i\pi/4}}_{z} - \underbrace{e^{-iat}e^{i\pi/4}}_{\bar{z}}}_{=2\,\mathrm{Im}(z)}\Big]$$

$z = [\cos(at) + i\sin(at)]\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$\Rightarrow 2\,\mathrm{Im}(z) = \sqrt{2}[-\cos(at) + \sin(at)]$

$$= \boxed{\frac{e^{at}}{2a} + \frac{1}{2a}\Big[\sin(at) - \cos(at)\Big]}$$

4. a) $f(z) = z \cosh \frac{1}{2z}$ $\qquad \left(\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \right)$

$$\Rightarrow f(z) = z \left[1 + \frac{1}{z^2 2^2 2!} + \frac{1}{z^4 2^4 4!} + \frac{1}{z^6 2^6 6!} + \cdots \right]$$

$$= z + \frac{1}{z\, 2^2 2!} + \frac{1}{z^3 2^4 4!} + \frac{1}{z^5 2^6 6!} + \cdots$$

i) singular pts are where the function fails to be analytic

$$\Rightarrow \boxed{z = 0 \text{ is a S.P}}$$

ii) The Laurent series about $z=0$ has an infinite number of negative powers of z

$$\Rightarrow \boxed{z = 0 \text{ is an essential S.P.}}$$

b) $f(z) = \frac{\sin 2z}{z^2}$

$$\Rightarrow f(z) = \frac{1}{z^2} \left[2z - \frac{2^3 z^3}{3!} + \frac{2^5 z^5}{5!} - \frac{2^7 z^7}{7!} + \cdots \right]$$

$$= \frac{2}{z} - \frac{2^3 z}{3!} + \frac{2^5 z^3}{5!} - + \cdots$$

i) $z = 0$ is a singular point

ii) From the Laurent series we see that $z=0$ is a simple pole

iii) $\underset{z=0}{\mathrm{Res}} \left[\frac{\sin 2z}{z^2} \right] = 2$

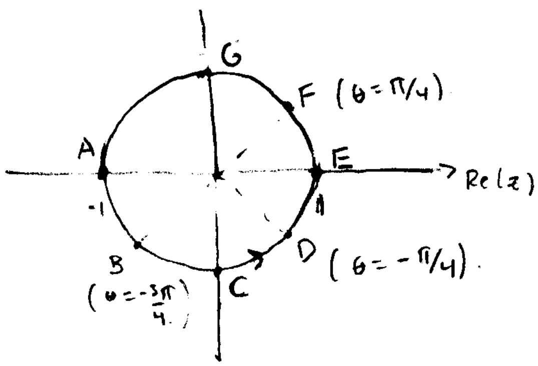
5. Mapping function is $w = f(z) = z + \frac{1}{z}$

Let $w = u + iv$ & $z = re^{i\theta}$

$$\Rightarrow W = re^{i\theta} + \frac{i}{r}e^{-i\theta} = r\cos\theta + ir\sin\theta + \frac{i}{r}\cos\theta + \frac{1}{r}\sin\theta$$

$$= \underbrace{r\cos\theta + \frac{1}{r}\sin\theta}_{u(r,\theta)} + i\underbrace{\left[r\sin\theta + \frac{1}{r}\cos\theta\right]}_{v(r,\theta)}$$

- Now, we are interested in the unit circle.



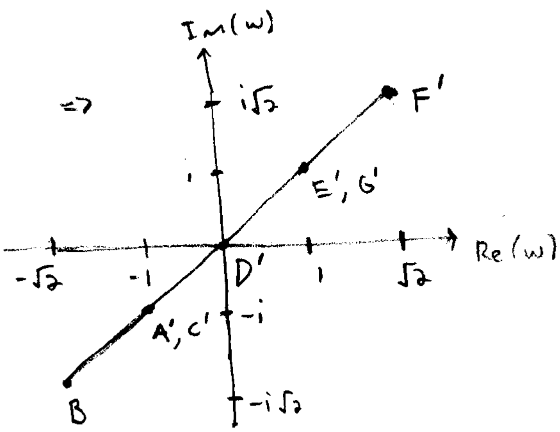
this curve is described by

$z = e^{i\theta}$ $(r = 1, -\pi \leq \theta \leq \pi)$

Thus, in the w-plane the curve becomes

$$W = \underbrace{\cos\theta + \sin\theta}_{u(1,\theta)} + i\underbrace{[\cos\theta + \sin\theta]}_{v(1,\theta)} \quad -\pi \leq \theta \leq \pi$$

$\Rightarrow$



6. $u(x,y) = \sinh x \sin y$

a) <u>Harmonic</u>

$u_x = \cosh x \sin y \qquad u_{xx} = \sinh x \sin y$

$u_y = \sinh x \cos y \qquad u_{yy} = -\sinh x \cos y$

$\Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic in the entire finite complex plane

<u>Conjugate:</u> Use Cauchy-Riemann equations.

$\Rightarrow v_y = u_x \quad \& \quad v_x = -u_y$

$\Rightarrow \dfrac{dv}{dy} = \cosh x \sin y \Rightarrow v = -\cosh x \cos y + \psi_1(x)$

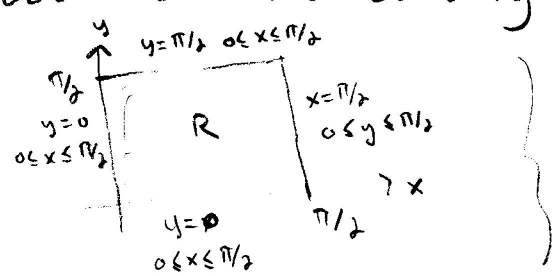
$\dfrac{dv}{dx} = -\sinh x \cos y \Rightarrow v = -\cosh x \cos y + \psi_2(y)$

$\Rightarrow \psi_1(x) = \psi_2 = \text{const.}$

Since we only need to find one conjugate set const $= 0$

$\Rightarrow \boxed{v(x,y) = -\cosh x \cos y}$

b) Since $u(x,y)$ is harmonic, we know its maximum value must occur on the boundary of the square



$\sinh x$ is maximum at $x = \pi/2$

$\sin y$ is maximum at $y = \pi/2$

$\Rightarrow$ Maximum value of u occurs at $x = \pi/2,\ y = \pi/2$

the maximum value is

$\boxed{u(\pi/2, \pi/2) = \sinh \pi/2}$

7. a) $f(z) = \dfrac{e^z - e}{z-1} \qquad |z-1| < \infty$

$\Rightarrow$ Expand about $z = 1$

$$f(z) = \frac{e(e^{z-1} - 1)}{z-1} = \frac{e}{z-1}\left[-1 + 1 + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}\right]$$

$$= e \sum_{n=1}^{\infty} \frac{(z-1)^{n-1}}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{e(z-1)^n}{(n+1)!}}$$

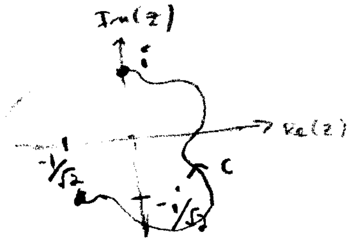
b) $f(z) = \dfrac{z}{a^2 - z^2}$, $|z| > a$ (Expand about $z = 0$)

$$= \frac{z}{-z^2\left(1 - \frac{a^2}{z^2}\right)} = \underbrace{-\frac{1}{z}\sum_{n=0}^{\infty} \frac{a^{2n}}{z^{2n}}}_{\text{valid for } \left|\frac{a^2}{z^2}\right| < 1 \Rightarrow |z| > a} = \boxed{-\sum_{n=0}^{\infty} \frac{a^{2n}}{z^{2n+1}}}$$

(8)

8.

a) $\int_c z^{1/5}\,dz$



$z^{1/5} = r^{1/5} e^{i\theta/5} \quad (r>0,\ \pi<\theta<3\pi)$

analytic in in the region for curve specified

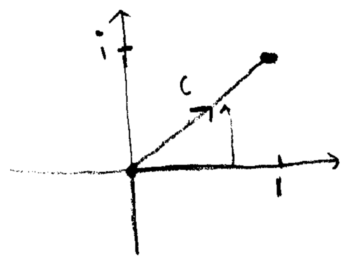
For the branch specified

$$\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = e^{i5\pi/4} \quad \& \quad i = e^{i5\pi/2}$$

$$\Rightarrow \int_c z^{1/5}\,dz = \frac{5}{6} z^{6/5}\Big]_{e^{i5\pi/4}}^{e^{i5\pi/2}} = \frac{5}{6}\left[e^{i3\pi} - e^{3\pi/2\, i}\right]$$

$$= \boxed{\frac{5}{6}\left[-1 + i\right]}$$

b) $\int_c \sin\bar{z}\,dz$



$C:\ z = x + ix \quad 0 \le x \le 1$

$= re^{i\pi/4} \quad 0 \le r \le \sqrt{2}$

$\Rightarrow dz = dr\, e^{i\pi/4}$

$\& \ \bar{z} = re^{-i\pi/4}$

$$\Rightarrow = \int_0^{\sqrt{2}} \sin(re^{-i\pi/4})\, e^{i\pi/4}\, dr$$

$$= e^{i\pi/4}\left[-e^{i\pi/4}\cos(re^{-i\pi/4})\right]_0^{\sqrt{2}} = \boxed{-i\left[\cos(1-i) - 1\right]}$$

9. Find roots of $\tanh w = \sqrt{2} + i$

$$\Rightarrow w = \tanh^{-1}(\sqrt{2}+i) = \frac{1}{2}\log\left(\frac{1+(\sqrt{2}+i)}{1-(\sqrt{2}+i)}\right) = \frac{1}{2}\log\left(\frac{1+\sqrt{2}+i}{1-\sqrt{2}-i}\right)$$

$$= \frac{1}{2}\log\left(\frac{(1+\sqrt{2}+i)(1-\sqrt{2}+i)}{(1-\sqrt{2})^2+1}\right) = \frac{1}{2}\log\left(\frac{-2+2i}{4-2\sqrt{2}}\right)$$

$$= \frac{1}{2}\log\left(\frac{-1+i}{2-\sqrt{2}}\right) = \frac{1}{2}\left[\ln\left|\frac{-1+i}{2-\sqrt{2}}\right| + i\left[\frac{3\pi}{4} + 2n\pi\right]\right]$$

$$= \boxed{\frac{1}{2}\ln(1+\sqrt{2}) + i\left[\frac{3\pi}{8} + n\pi\right] \quad n = 0, \pm 1, \pm 2, \ldots}$$