

1. In this exercise you will be comparing several different quadrature methods for approximating the following integrals:

$$I(f_1) = \int_{-1}^1 (x-1)^2 e^{-x^2} dx,$$

$$I(f_2) = 2 \int_{-1}^1 \frac{1}{1+x^2} dx,$$

$$I(f_3) = \int_{-1}^1 e^{\cos \pi x} dx = 2.53213175550401667 \dots$$

Please answer all parts of the exercises a–c below.

- (a) Use the symbolic tool box to obtain exact answers to $I(f_1)$ and $I(f_2)$.
- (b) Using composite Midpoint rule, compute an approximation to $I(f_1)$, $I(f_2)$, and $I(f_3)$ for $n = 4, 8, 16, 32, 64, 128$. (i) Report the error in these approximations for each n in a nice table and produce a plot of the error vs. n (on a log-log scale). (ii) For the $I(f_1)$ and $I(f_2)$ results, verify that the error is decreasing like $O(h^2)$. (iii) Does this rate of decrease in the error appear to be true for $I(f_3)$?
- (c) Repeat part (b) for composite Trapezoidal rule.
- (d) Repeat part (b) for the composite Simpson's rule. For part (ii) the error rate should be $O(h^4)$.
- (e) Compare the computational cost (in terms of function evaluations) vs. accuracy of the three methods above to the book's adaptive Simpson's rule function `quadtx`. You should set the tolerance in the `quadtx` function to the error produced by the three functions above for an appropriate value of n and compare the function evaluations of the adaptive method to the function evaluations of the plain composite method.

Note that for $I(f_3)$, midpoint and trapezoidal rule should be extremely accurate. This is not a fluke. For smooth periodic integrands that are integrated over their period, these methods will outperform nearly every other method. The reason for this should be explained in more advanced numerical analysis courses.

2. Midpoint, trapezoidal, and Simpson's rule discretize the integrand with equally spaced nodes and then linearly combine the function values at these nodes in order to approximate the integral to some order of accuracy. Gaussian quadrature also uses a linear combination of values of the integrand to approximate the integral, but instead of using equally spaced nodes, it uses non-equally spaced nodes so that the linear combination of function values at these nodes *maximizes* the order of accuracy that can be achieved. For example, the nodes and weights for the Gaussian quadrature formula with $n = 9$ are given by

x_i	c_i
-0.96816023950763	0.081274388361574
-0.83603110732664	0.18064816069486
-0.61337143270059	0.26061069640294
-0.32425342340381	0.31234707704
0	0.33023935500126
0.32425342340381	0.31234707704
0.61337143270059	0.26061069640294
0.83603110732664	0.18064816069486
0.96816023950763	0.081274388361574

These nodes and weights are used as follows:

$$\int_{-1}^1 f(x)dx = \sum_{i=1}^9 c_i f(x_i)$$

For sufficiently smooth f , this formula has an order of 18, making it significantly more accurate than Simpson's rule.

- Compute an approximation to $I(f_1)$, $I(f_2)$, and $I(f_3)$ from the previous problem. Report the error in the approximations in a nice table.
- Compare the errors you obtained through Gaussian quadrature for these three integrals with the results from (b), (c), (d), and (e) above. Again, you should look at error vs. number of function evaluations. What, if any, conclusions can be drawn.
- MATLAB's `quadl` function uses adaptive Gaussian quadrature to approximate integrals, whereas the `quad` function uses adaptive Simpson's rule. Use `quadl` to approximate $I(f_1)$, $I(f_2)$, and $I(f_3)$ with an error tolerance of 10^{-10} . Repeat this exercise for the `quad` function. Compare the number of function evaluations each of these routines takes to achieve the error tolerance for each integral.

You can obtain the number of function evaluations from `quadl` with the function call:

```
[q,fnct] = quadl(f,a,b,1e-10)
```

This also works for the `quad` function.

3. NCM, problem 6.12

- Trapezoidal rule is based on approximating the integrand $f(x)$ with a linear polynomial in each subinterval and then integrating it exactly. If the derivative of the integrand is also available at each subinterval then we can improve upon the Trapezoidal rule by approximating the integrand with a cubic Hermite polynomial in each subinterval and integrating it exactly. Recall that the cubic Hermite polynomial in the interval $[x_k, x_{k+1}]$ is given by

$$p(s) = \frac{h_k^3 - 3h_k s^2 + 2s^3}{h_k^3} f_k + \frac{3h_k s^2 - 2s^3}{h_k^3} f_{k+1} + \frac{s(s-h_k)^2}{h_k^2} f'_k + \frac{s^2(s-h_k)}{h_k^2} f'_{k+1},$$

where $s = x - x_k$, and $h_k = x_{k+1} - x_k$.

- Use this formula to derive the quadrature formula:

$$\int_{x_k}^{x_{k+1}} f(x)dx = \int_0^{h_k} f(s)ds \approx \frac{h_k}{2} [f_k + f_{k+1}] + \frac{h_k^2}{12} [f'_k - f'_{k+1}]$$

- The cubic Hermite quadrature formula above can be used as a composite method as follows:

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n-1} \left(\frac{h_k}{2} [f_k + f_{k+1}] + \frac{h_k^2}{12} [f'_k - f'_{k+1}] \right),$$

where $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$. If the nodes x_k , $k = 0, 1, \dots, n$ are equally spaced with spacing h (i.e. $h = (b - a)/n$, and $x_k = a + kh$), then this composite formula reduces to the simple formula

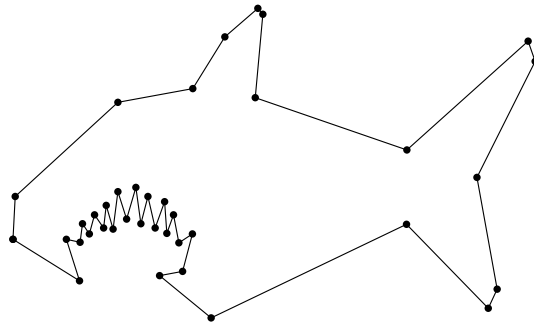
$$\int_a^b f(x)dx \approx \left(h \sum_{k=1}^{n-1} f_k \right) + \frac{h}{2}[f_0 + f_n] + \frac{h^2}{12}[f'_0 - f'_n],$$

Determine the order of accuracy of this composite formula, which is sometimes called the “corrected Trapezoidal rule”.

- (c) Repeat part (b) from problem 1 for the corrected Trapezoid above, but only for $I(f_1)$. Compare the results to the regular composite Trapezoidal rule.

5. NCM, problem 6.21

6. In this exercise you will be determining the area of the shark figure from homework 3 (shown below), which you can download from the course web page.



- (a) The `shark.mat` file contains a list of coordinates, (x_k, y_k) , $k = 1, 2, \dots, n$, that define the outline of the shark and can be used to parametrically interpolate the shark. These coordinates can be viewed as vertices of a polygon defining the shark since none of the lines connecting the vertices intersect. A classic formula for determining the area of polygon is

$$\text{Area} = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \dots + x_ny_1 - x_1y_n)$$

Use this formula to estimate the area of the shark.

- (b) A much more accurate way to determine the area of the shark is to first construct a parametric interpolant to the coordinates (x_k, y_k) , $k = 1, 2, \dots, n$, as you did in problem 5(c) from homework 3 with the `splintx` function. This gives you functions $x(t)$ and $y(t)$, such that $(x(k), y(k)) = (x_k, y_k)$, $k = 1, 2, \dots, n$. The second step is to use the formula from calculus for the area enclosed by a closed parametric curve

$$\text{Area} = \int_a^b x(t)y'(t)dt,$$

where $a = 1$ and $b = n$ for this example.

Using the `splintx` function construct a parametric interpolant to the shark coordinates. Then, use the `quadt` function to approximate the integral above and get an estimate for the area. Compare the results from (a) and (b) and turn in a listing of your code.

7. *Plane of best fit.* Given experimental data $(x_j, y_j, f(x_j, y_j))$, $j = 1, 2, \dots, n$, the plane $p(x, y) = a + bx + cy$ that best fits this data in the least-squares sense, can be computed with the backslash operator `\` in MATLAB (which uses *QR* decomposition) as discussed in class. You will use this technique for the following problem:

Employees at a leading shoe store were given a psychological exam to measure their intelligence (a score on a scale of 50-low intelligence to 150-high intelligence) and extroversion (a score on

a scale of 15-low extroversion to 30-high extroversion). The company then compared these two numbers with the average amount of sales each employee generated per week. The data is given in the table below:

Sales Person	Intelligence	Extroversion	Sales/Week
1	89	21	2625
2	93	24	2700
3	91	21	3100
4	122	23	3150
5	115	27	3175
6	100	18	3100
7	98	19	2700
8	105	16	2475
9	112	23	3625
10	109	28	3525

Determine the plane of best fit to this data and use it to predict the average weekly sales of an employee with an intelligence score of 150 and an extroversion score of 18.