Math 175-004 **In Class Assignment 4** **Mar. 31, 2008**

Names:____

For this assignment you will be determining whether a given series converges or diverges using the geometric series test, n^{th}-term test for divergence, the integeral test, or the comparison test. The latter test consists of applying the following result:

- Suppose $a_n \geq 0$, for all $n \geq N$, where N is some integer. Then
  (a) $\displaystyle\sum_{n=N}^{\infty} a_n$ converges if there is a convergent series $\displaystyle\sum_{n=N}^{\infty} c_n$ with $a_n \leq c_n$ for all $n > N$.
  (b) $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges if there is a divergent series of nonnegative terms $\displaystyle\sum_{n=N}^{\infty} d_n$ with $a_n \geq d_n$ for all $n > N$.

The following examples illustrate how to apply this result:

1. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converges since $\displaystyle\frac{1}{n^2 \ln n} \leq \frac{1}{n^2}$ for $n \geq 3$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges by the p-series test.

2. $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges since $\displaystyle\frac{\ln n}{n} \geq \frac{1}{n}$ for $n \geq 3$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (it is the harmonic series).

3. $\displaystyle 7 + 3 - 1 + 4 + \sum_{n=2}^{\infty} \frac{1}{n^2 + \sqrt{n}}$ converges since $\displaystyle\frac{1}{n^2 + \sqrt{n}} \leq \frac{1}{n^2}$ for $n \geq 2$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p-series test.

Another helpful result from the comparison test is the following:

- Suppose $a_n \geq 0$, for all $n \geq N$, where N is some integer. Then
  (a) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\displaystyle\sum_{n=N}^{\infty} a_n$ and $\displaystyle\sum_{n=N}^{\infty} b_n$ both converge or both diverge.
  (b) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\displaystyle\sum_{n=N}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=N}^{\infty} a_n$ converges.
  (b) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\displaystyle\sum_{n=N}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges.

The following examples illustrate how to apply this result:

1. $\displaystyle\sum_{n=1}^{\infty} \tan\frac{1}{n}$ diverges since we can let $b_n = \dfrac{1}{n}$ and note that

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\tan\frac{1}{n}}{\frac{1}{n}} && \text{(apply L'Hospital's rule)}\\
&= \lim_{n\to\infty}\frac{-\frac{1}{n^2}\sec^2\frac{1}{n}}{-\frac{1}{n^2}}\\
&= \lim_{n\to\infty}\sec^2\frac{1}{n} = 1 > 0
\end{aligned}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$ converges since we can let $b_n = \dfrac{1}{n^{\frac{3}{2}}}$ and note that

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{\frac{3}{2}}}}\\
&= \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1 > 0
\end{aligned}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges by the p-series test.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(1+\ln n)^2}$ diverges since we can let $b_n = \dfrac{1}{n}$ and note that

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{1}{(1+\ln n)^2}}{\frac{1}{n}}\\
&= \lim_{n\to\infty}\frac{n}{(1+\ln n)^2} && \text{(apply L'Hospital's rule)}\\
&= \lim_{n\to\infty}\frac{1}{\frac{2}{n}(1+\ln n)}\\
&= \lim_{n\to\infty}\frac{n}{2(1+\ln n)} && \text{(apply L'Hospital's rule)}\\
&= \lim_{n\to\infty}\frac{1}{\frac{2}{n}} = \infty
\end{aligned}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Problems: Use the geometric series test, n^{th}-term test for divergence, the integeral test, or the comparison test to determine if the following series converge or diverge. Show your work.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^4 n}{\sqrt{n}n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n^2 - 5}$

4. $\sum_{n=1}^{\infty} \frac{3n^2 - 2n}{n^4 + n^2 + 1}$

5. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$

6. $\sum_{n=1}^{\infty} \frac{\ln n}{\tanh n}$