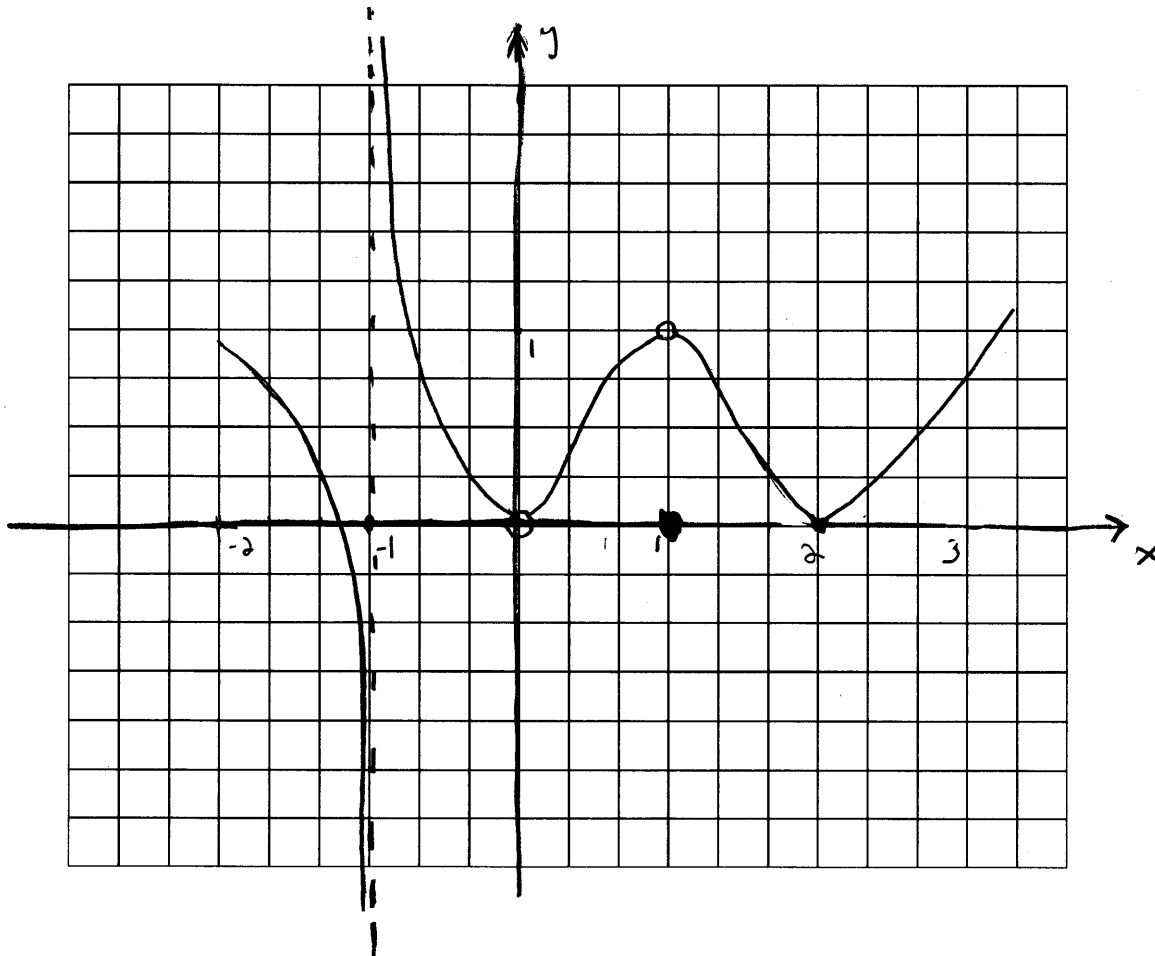


1. (Limits, continuity, and differentiability) Sketch the graph of a function $f(x)$ with the following properties:

- (a) f is continuous on the intervals $[-2, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, 3]$;
- (b) f is discontinuous at $x = 0$, $f(0)$ does not exist, but $\lim_{x \rightarrow 0} f(x) = 0$;
- (c) $\lim_{x \rightarrow -1^-} f(x) = -\infty$ and $\lim_{x \rightarrow -1^+} f(x) = \infty$;
- (d) f is discontinuous at $x = 1$, $f(1) = 0$, but $\lim_{x \rightarrow 1} f(x) = 1$;
- (e) f is continuous at $x = 2$, but not differentiable there.

Label the plot so that the above requirements are obvious.



2. (Definite integrals) Evaluate the following

$$(a) \int_{-1}^1 (10y^4 + 8y^3 - 12y^2 - 2y + 5) dy$$

$$= 2 \int_0^1 (10y^4 - 12y^2 + 5) dy \quad (\text{use odd symmetry property})$$

$$= 2 \left[2y^5 - 4y^3 + 5y \right]_0^1$$

$$= 2 [2 - 4 + 5] = 6$$

$$(b) \int_{-1}^1 \frac{e^{1/x}}{x^2} dx \quad u = \frac{1}{x} \quad \frac{du}{dx} = -\frac{1}{x^2} \quad \text{or} \quad -du = \frac{dx}{x^2}$$

$$\Rightarrow -\int_{-1}^1 e^u du = -e^u \Big|_{-1}^1 = -e + e^{-1} = \frac{1}{e} - e$$

$$(c) \int_0^{\ln 2} \frac{\sinh x}{\sqrt{1 + \cosh x}} dx \quad (\text{simplify your answer}) \quad u = \underbrace{1 + \cosh x}_{g(x)}, \quad \frac{du}{dx} = \sinh x$$

$$g(0) = 2, \quad g(\ln 2) = 1 + \cosh(\ln 2) = 1 + \frac{1}{2} \left(2 + \frac{1}{2} \right) = 2 + \frac{1}{4} = \frac{9}{4}$$

$$\int_2^{9/4} \frac{du}{\sqrt{u}} = \int_2^{9/4} u^{-1/2} du = \left[2u^{1/2} \right]_2^{9/4} = 2 \left[\frac{3}{2} + \sqrt{2} \right] = \boxed{3 + 2\sqrt{2}}$$

3. (Fundamental theorem of calculus)

(a) Find dy/dx if $y = \int_1^{e^x} \frac{1}{t^2} dt$ Let $u = e^x$, $\frac{du}{dx} = e^x$

Then $y(u) = \int_1^u \frac{1}{t^2} dt$ & $\frac{dy}{du} = \frac{1}{u^2}$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u^2} \cdot e^x = \frac{e^x}{e^{2x}} = \frac{1}{e^x} = e^{-x}$

(b) Determine the following limit: $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x (1 - \cos^2 t) dt$

$$\lim_{x \rightarrow 0} \frac{\int_0^x (1 - \cos^2 t) dt}{x^3} = \frac{0}{0}$$

L'Hospital $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{3x^2} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{3x^2} = \frac{1}{3} \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 = \frac{1}{3}$

5. (Initial value problem, 20 pts) A particle moves along the x -axis with a velocity of $v(t) = te^{-t^2} + 1$. Assuming that the velocity of the particle at time $t = 0$ is $s = 1/2$, find

(a) An equation for the position in terms of t .

$$S(t) = \int v(t) dt = \int (te^{-t^2} + 1) dt = \int te^{-t^2} dt + \int dt$$

$$u = t^2, \quad \frac{du}{dt} = 2t \quad \Rightarrow \quad \frac{du}{2} = t dt$$

$$S(t) = \frac{1}{2} \int (e^{-u} du) + \int dt = \frac{1}{2} (-e^{-u} + t + C)$$

$$= -\frac{1}{2} e^{-t^2} + t + C$$

$$S(0) = -\frac{1}{2} + C = \frac{1}{2} \quad \Rightarrow \quad C = 1 \quad \Rightarrow \quad \boxed{S(t) = -\frac{1}{2} e^{-t^2} + t + 1}$$

(b) An equation for the acceleration in terms of t .

$$a(t) = v'(t) = \frac{d}{dt} (te^{-t^2} + 1)$$

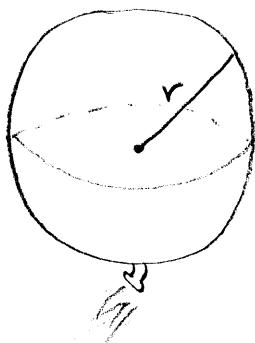
$$= e^{-t^2} - 2t^2 e^{-t^2}$$

6. **(Related rates)** A spherical balloon filled with helium is deflating in such a way that its surface area is decreasing at rate of $32 \frac{\text{ft}^2}{\text{min}}$.

(a) At what rate is the balloon's radius changing the instant the radius is 4 ft?

(b) How is the volume changing?

Note the volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the surface area is $S = 4\pi r^2$.



$$\frac{ds}{dt} = -32 \frac{\text{ft}^2}{\text{min}}$$

$$- S(t) = 4\pi [r(t)]^2$$

$$- \text{Find } \frac{dr}{dt}$$

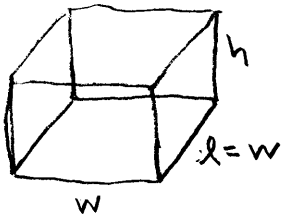
$$a) \frac{ds}{dt} = 8\pi r(t) \cdot r'(t) = -32 \frac{\text{ft}^2}{\text{min}}$$

$$\Rightarrow r'(t) = \frac{-32}{8\pi r(t)} = -\frac{4}{\pi \cdot 4} = \boxed{-\frac{1}{\pi} \text{ft}}$$

$$b) V(t) = \frac{4}{3}\pi [r(t)]^3 \Rightarrow V'(t) = 4\pi [r(t)]^2 r'(t)$$

$$\Rightarrow V'(t) = 4\pi [4]^2 \cdot \left(-\frac{1}{\pi}\right) = \boxed{-64 \frac{\text{ft}^3}{\text{min}}}$$

7. (Optimization, 20 pts) We want to construct a rectangular box with a square base and we only have 54 ft^2 of material to use in construction (the box is not open on any of its sides). Assuming that all the material is used in the construction process determine the maximum volume that the box can have. Note that the volume of a rectangular box with width w , length l , and height h is $V = wlh$ and the surface area is $A = 2hl + 2wh + 2wl$. (Hint: First find an equation relating the height of the box to the width).



$$\begin{aligned} A &= 2hl + 2wh + 2wl \\ &= 2hw + 2wh + 2w^2 \\ &= 4hw + 2w^2 = 54 \\ \Rightarrow h &= \frac{27}{2w} - \frac{w}{2} \end{aligned}$$

$$V = wlh = w^2 h = w^2 \left(\frac{27}{2w} - \frac{w}{2} \right) = \frac{27}{2} w - \frac{w^3}{2}$$

Find maximum volume.

$$\frac{dV}{dw} = \frac{27}{2} - \frac{3}{2} w^2 = 0 \Rightarrow w^2 = 9$$

C.P. $w = \pm 3$ only $w = 3$ makes sense.

Check that it's a maximum:

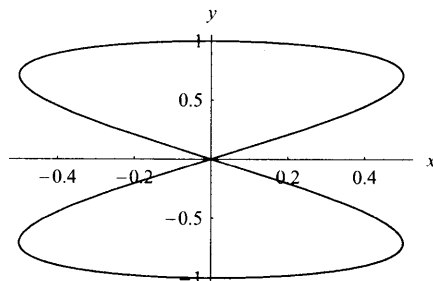
$$\frac{d^2V}{dw^2} = -3w, \quad \left. \frac{d^2V}{dw^2} \right|_{w=3} = -9 < 0 \quad \text{so } V'' < 0$$

Thus, $w = 3$ is a max

Dimensions: $w = l = 3 \text{ ft.}$ $h = \frac{27}{2 \cdot 3} - \frac{3}{2} = 3 \text{ ft.}$

8. (Implicit differentiation and ~~Parametric curves~~)

- (a) The curve given in the figure below is defined implicitly by the equation $y^4 = y^2 - x^2$. Determine the slope of the curve at the point $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$.



Compute derivative of $y^4 = y^2 - x^2$ with respect to x

$$\Rightarrow 4y^3 \frac{dy}{dx} = 2y \frac{dy}{dx} - 2x$$

Now plug in $x = \frac{\sqrt{3}}{4}$, $y = \frac{1}{2}$

$$4\left(\frac{1}{2}\right)^3 \frac{dy}{dx} = 2\left(\frac{1}{2}\right) \frac{dy}{dx} - 2\left(\frac{\sqrt{3}}{4}\right)$$

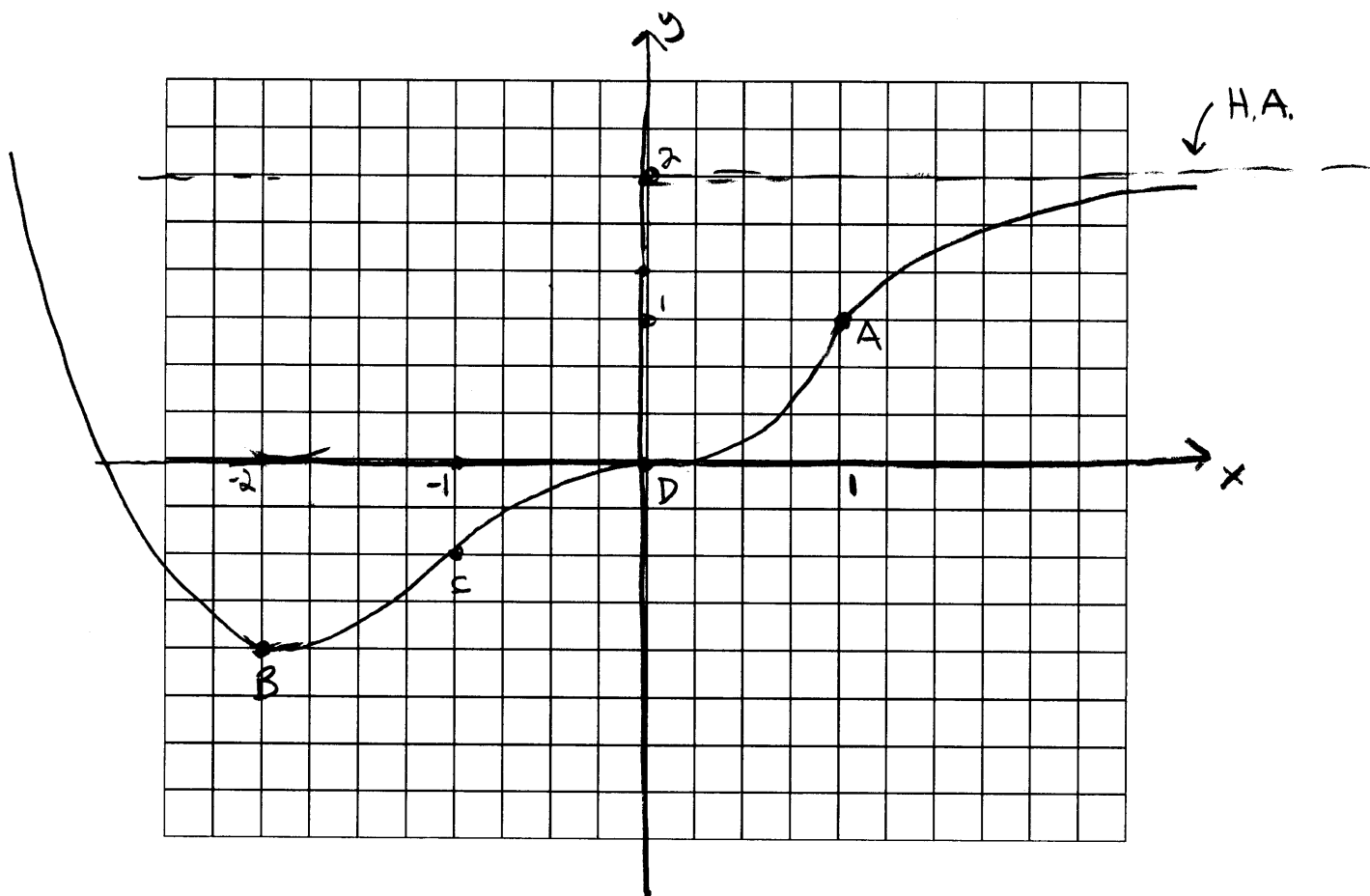
$$\frac{1}{2} \frac{dy}{dx} - \frac{dy}{dx} = -2\sqrt{3}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \sqrt{3}}$$

9. (Curve sketching) Sketch the graph of a function $f(x)$ that satisfies all of the following conditions.

- (a) $f(0) = 0$ and $f(1) = 1$;
- (b) $f'(-2) = 0$ and $f'(0) = 0$;
- (c) $f''(1) = 0$;
- (d) $f'(x) < 0$ for $-\infty < x < -2$ and $f'(x) > 0$ for $0 < x < \infty$;
- (e) $f''(x) > 0$ for $-2 \leq x < -1$, $f''(x) < 0$ for $-1 < x < 0$, and $f''(x) > 0$ for $0 < x < 1$;
- (f) $\lim_{x \rightarrow \infty} f(x) = 2$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$;

Label coordinate axes, all extreme points, inflection points, x -intercepts, and y -intercepts, vertical asymptotes, and horizontal asymptotes.



- A = pt. of inflection
- B = global min
- C = pt of inflection
- D = pt of inflection