Chapter 4 Exponential and Logarithmic Functions

4.5 Modeling with Exponential and Logarithmic Functions

Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, and numerous others, can be modeled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena. In this section we study exponential and logarithmic models.

Problems: 1, 3, 5, **7, 11, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 39, 41**
These are the minimal problems you should work to gain an understanding of the material in this section.

Terms: exponential growth model, exponential growth, population growth, radioactive decay, half-life, radioactive decay model, Newton's Law of Cooling, logarithmic scales, pH scale, Richter scale, magnitude, intensity, decibel scale, intensity level

You should know the definitions of these terms.

Concepts, Processes, Techniques:

solve population growth problems,
solve radioactive decay problems,
solve Newton's Law of Cooling problems,
solve logarithmic scale problems,
solve pH scale problems,
solve Richter scale problems,
solve decibel scale problems

These are the concepts, processes and techniques that you are trying to gain a mastery of by working the problems and understanding the material.

When reading the section work through:

Solve exponential growth problems
Solve population growth problems: like Examples 1-5 p 370-3
Solve radioactive decay problems: like Example 6 p 374
Solve Newton's Law of Cooling: like Example 7 p 375
Solve logarithmic scale problems
Solve pH scale problems: like Example 8 p 377
Solve Richter scale problems: like Examples 9-10 p 378
Solve decibel scale problems: like Examples 11 p 379

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Exponential Growth Model

A population that experiences **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where $n(t)$ = population at time t

n_0 = initial size of the population

r = relative rate of growth (expressed as a proportion of the population)

t = time

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. Using calculus, the following model can be deduced from this law.

Newton's Law of Cooling

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s, then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

where k is a positive constant that depends on the type of object.

Logarithmic Scales

When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers. We discuss three such situations: the pH scale, which measures acidity; the Richter scale, which measures the intensity of earthquakes; and the decibel scale, which measures the loudness of sounds. Other quantities that are measured on logarithmic scales are light intensity, information capacity, and radiation.

THE pH SCALE Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Sorensen, in 1909, proposed a more convenient measure. He defined

$$\text{pH} = -\log[\text{H}^+]$$

THE RICHTER SCALE In 1935 the American geologist Charles Richter (1900–1984) defined the magnitude M of an earthquake to be

$$M = \log \frac{I}{S}$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard" earthquake (whose amplitude is 1 micron $= 10^{-4}$ cm). The magnitude of a standard earthquake is

$$M = \log \frac{S}{S} = \log 1 = 0$$

THE DECIBEL SCALE The ear is sensitive to an extremely wide range of sound intensities. We take as a reference intensity $I_0 = 10^{-12}$ W/m^2 (watts per square meter) at a frequency of 1000 hertz, which measures a sound that is just barely audible (the threshold of hearing). The psychological sensation of loudness varies with the logarithm of the intensity (the Weber-Fechner Law) and so the **intensity level** B, measured in decibels (dB), is defined as

$$B = 10 \log \frac{I}{I_0}$$

The intensity level of the barely audible reference sound is

$$B = 10 \log \frac{I_0}{I_0} = 10 \log 1 = 0 \text{ dB}$$